Midterm **Nov. 8, 2006** **Physics 130B**

1. At $t = 0$, an electron is the spin state

$$\psi(t=0) = \begin{pmatrix} i\sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}.$$

A magnetic field B is applied in the z direction. **a)** Find the spin state of the particle as a function of time. **b)** Find the expectation value of S_y as a function of time. **c)** What is the probability to measure that the electron's spin along the x direction is $\frac{\hbar}{2}$ as a function of time? *(10 points)*

$$H = -\vec{\mu} \cdot \vec{B} = \mu_B B \sigma_z$$

a) The eigenstates are then spin up and spin down along the z direction since these are the eigenstates of σ_z. The energies for spin up and down along the z direction are $\pm\mu_B B$. So it is easy to write $\psi(t)$ since the two components of the spinor represent the energy eigenstates. Let $\omega = \frac{\mu_B B}{\hbar}$.

$$\psi(t) = \begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix}.$$

b) Simply compute the expectation value in this state.

$$\langle S_y \rangle = \frac{\hbar}{2}\begin{pmatrix} -i\sqrt{\frac{2}{3}}e^{i\omega t} & \sqrt{\frac{1}{3}}e^{-i\omega t} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -i\sqrt{\frac{2}{3}}e^{i\omega t} & \sqrt{\frac{1}{3}}e^{-i\omega t} \end{pmatrix}\begin{pmatrix} -i\sqrt{\frac{1}{3}}e^{i\omega t} \\ -\sqrt{\frac{2}{3}}e^{-i\omega t} \end{pmatrix}$$

$$\langle S_y \rangle = \frac{\hbar}{2}\left[-\frac{\sqrt{2}}{3}e^{i2\omega t} - \frac{\sqrt{2}}{3}e^{-i2\omega t}\right] = -\frac{\sqrt{2}}{3}\frac{\hbar}{2}2\cos(2\omega t) = -\frac{\sqrt{2}}{3}\hbar\cos(2\omega t)$$

c) To get the amplitude to have spin up in the x direction, dot with that eigenstate $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$$P = \left|\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix}\right|^2 = \left|i\sqrt{\frac{1}{3}}e^{-i\omega t} + \sqrt{\frac{1}{6}}e^{i\omega t}\right|^2 = \frac{1}{3} + \frac{1}{6} + i\sqrt{\frac{1}{3}}\sqrt{\frac{1}{6}}e^{-2i\omega t} - i\sqrt{\frac{1}{3}}\sqrt{\frac{1}{6}}e^{2i\omega t}$$

$$= \frac{1}{2} + \sqrt{\frac{1}{18}}2\sin(2\omega t) = \frac{1}{2} + \sqrt{\frac{2}{9}}\sin(2\omega t)$$

2. Consider a two dimensional harmonic oscillator problem described by the Hamiltonian

$$H_0 = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2).$$

Remember that this unperturbed problem can be easily solved in Cartesian coordinates. **a)** Accurately calculate the energy shifts to the degenerate first excited states, to first order, if the additional potential $V = \beta xy$ is applied. **b)** What are the new energy eigenstates for the degenerate first excited state (in terms of the standard eigenstates $u_0^{(x)}u_1^{(y)}$ and $u_1^{(x)}u_0^{(y)}$)? *(10 points)*

As stated in the problem, the degenerate first excited states are $u_1^{(x)}u_0^{(y)} = |10\rangle$ and $u_0^{(x)}u_1^{(y)} = |01\rangle$. The excitation in x has the same energy as the excitation in y. The perturbation V can be written in terms of the raising and lowering operators for x and y (which are not the same thing).

$$V = \beta xy = \frac{\beta\hbar}{2m\omega}\left(A + A^\dagger\right)_x \left(A + A^\dagger\right)_y$$

We need to compute the 2 by 2 matrix for V for these two states. The only non-zero matrix elements are

$$\langle 01|V|10\rangle = \langle 10|V|01\rangle = \frac{\beta\hbar}{2m\omega}\langle 01|A_{(x)}A_{(y)}^\dagger|10\rangle = \frac{\beta\hbar}{2m\omega}$$

$E_\pm = 2\hbar\omega \pm \frac{\beta\hbar}{2m\omega}$ for the energy eigenstates $\psi_\pm = \frac{1}{\sqrt{2}}(u_0u_1 \pm u_1u_0)$.

3. Two non-identical spin $\frac{1}{2}$ particles have a Hamiltonian given by

$$H = E_0 + B\vec{S}_1 \cdot \vec{S}_2 + C(S_{1z} + S_{2z}).$$

There are no spatial coordinates for this problem. Find the allowed energies and the energy eigenstates in terms of the four product states $\chi_+^{(1)}\chi_+^{(2)}$, $\chi_+^{(1)}\chi_-^{(2)}$, $\chi_-^{(1)}\chi_+^{(2)}$, and $\chi_-^{(1)}\chi_-^{(2)}$. *(10 points)*

The eigenstates of the the total spin operator S^2 will be the eigenstates of this Hamiltonian since $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$. The energies are $E = E_0 + B\frac{\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right) + C\hbar m_s$ for the states $|sm_s\rangle$.

$$
\begin{aligned}
|11\rangle &= \chi_+\chi_+ \quad && E_0 + \tfrac{B\hbar^2}{4} + C\hbar \\
|1\,-1\rangle &= \chi_-\chi_- \quad && E_0 + \tfrac{B\hbar^2}{4} - C\hbar \\
|10\rangle &= \frac{1}{\sqrt{2}}(\chi_+\chi_- + \chi_-\chi_+) \quad && E_0 + \tfrac{B\hbar^2}{4} \\
|00\rangle &= \frac{1}{\sqrt{2}}(\chi_+\chi_- - \chi_-\chi_+) \quad && E_0 - \tfrac{3B\hbar^2}{4}
\end{aligned}
$$

$\hbar = 1.05 \times 10^{-27}$ erg sec	$c = 3.00 \times 10^{10}$ cm/sec	$e = 1.602 \times 10^{-19}$ coulomb
$1eV = 1.602 \times 10^{-12}$ erg	$\alpha = \frac{e^2}{\hbar c} = 1/137$	$\hbar c = 1973$ eV Å$= 197.3$ MeV F
1 Å$= 1.0 \times 10^{-8}$ cm	1 Fermi $= 1.0 \times 10^{-13}$ cm	$a_0 = \frac{\hbar}{\alpha m_e c} = 0.529 \times 10^{-8}$ cm
$m_p = 938.3$ MeV/c^2	$m_n = 939.6$ MeV/c^2	$m_e = 9.11 \times 10^{-28}$ g $= 0.511$ MeV/c^2
$k_B = 1.38 \times 10^{-16}$ erg/°K	$g_e = 2 + \frac{\alpha}{\pi}$	$g_p = 5.6$
$\mu_{Bohr} = \frac{e\hbar}{2m_e c} = 0.579 \times 10^{-8}$ eV/gauss		$\int\limits_{-\infty}^{\infty} dx\ f(x)\ \delta(g(x)) = \left[\frac{1}{\lvert\frac{dg}{dx}\rvert} f(x)\right]_{g(x)=0}$
$\int\limits_{-\infty}^{\infty} dx\ f(x)\ \delta(x-a) = f(a)$	$\int\limits_{-\infty}^{\infty} dx\ e^{-ax^2} = \sqrt{\frac{\pi}{a}}$	use $\frac{\partial}{\partial a}$ for other forms
$e^A = \sum\limits_{n=0}^{\infty} \frac{A^n}{n!}$	$\sin\theta = \sum\limits_{n=1,3,5...}^{\infty} \frac{\theta^n}{n!}(-1)^{\frac{n-1}{2}}$	$\cos\theta = \sum\limits_{n=0,2,4...}^{\infty} \frac{\theta^n}{n!}(-1)^{\frac{n}{2}}$
$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$	$\int\limits_{0}^{\infty} dr\ r^n e^{-ar} = \frac{n!}{a^{n+1}}$	$E = \sqrt{m^2c^4 + p^2c^2}$

HARMONIC OSCILLATOR

$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega A^\dagger A + \frac{1}{2}\hbar\omega$		$E_n = (n + \frac{1}{2})\hbar\omega \quad n = 0, 1, 2...$
$u_n(x) = \sum\limits_{k=0}^{\infty} a_k y^k e^{-y^2/2}$	$a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k$	$y = \sqrt{\frac{m\omega}{\hbar}} x$
$A = (\sqrt{\frac{m\omega}{2\hbar}} x + i\frac{p}{\sqrt{2m\hbar\omega}})$	$A^\dagger = (\sqrt{\frac{m\omega}{2\hbar}} x - i\frac{p}{\sqrt{2m\hbar\omega}})$	$[A, A^\dagger] = 1$
$A^\dagger \lvert n\rangle = \sqrt{(n+1)}\ \lvert n+1\rangle$	$A \lvert n\rangle = \sqrt{(n)}\ \lvert n-1\rangle$	$u_0(x) = (\frac{m\omega}{\hbar\pi})^{\frac{1}{4}}\ e^{-m\omega x^2/2\hbar}$

GENERAL WAVE MECHANICS

$E = h\nu = \hbar\omega$	$\lambda = h/p$	$p = \hbar k$
$\Delta p\ \Delta x \geq \frac{\hbar}{2}$	$\Delta A\ \Delta B \geq \langle\frac{1}{2}[A, B]\rangle$	$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$
$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} dp\ \phi(p)\ e^{ipx/\hbar}$		$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} dx\ \psi(x)\ e^{-ipx/\hbar}$
$p_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$	$E_{op} = i\hbar\frac{\partial}{\partial t}$	$x_{op} = i\hbar\frac{\partial}{\partial p}$
$Hu_j(x) = E_j u_j(x)$	$\psi_j(x, t) = u_j(x)e^{-iE_jt/\hbar}$	$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$
$\psi(x)$ continuous	$\frac{d\psi}{dx}$ continous if V finite	
$\Delta\frac{d\psi}{dx} = \frac{2m\lambda}{\hbar^2}\psi(a)$ for $V(x) = \lambda\delta(x-a)$		
$\langle\phi\vert\psi\rangle = \int\limits_{-\infty}^{\infty} dx\phi^*(x)\psi(x)$	$\langle u_i\vert u_j\rangle = \delta_{ij}$	$\sum\limits_i \lvert u_i\rangle\langle u_i\rvert = 1$
$\phi = \sum\limits_i a_i u_i$	$a_i = \langle u_i\vert\phi\rangle$	$\psi(x) = \langle x\vert\psi\rangle$
$\langle\phi\vert A\vert\psi\rangle = \langle\psi\vert A\vert\phi\rangle^* = \langle\phi\vert A\psi\rangle = \langle A^\dagger\phi\vert\psi\rangle$		$\phi(p) = \langle p\vert\psi\rangle$
$[\frac{1}{2m}(\vec{p} + \frac{e}{c}\vec{A})^2 + V(\vec{r})]\psi(\vec{r}) = E\psi(\vec{r})$		$H\psi = E\psi$
$[p_x, x] = \frac{\hbar}{i}$	$[L_x, L_y] = i\hbar L_z$	$[L^2, L_z] = 0$
$\psi_i = \langle u_i\vert\psi\rangle$	$A_{ij} = \langle u_i\vert A\vert u_j\rangle$	$\frac{d\langle A\rangle}{dt} = \langle\frac{\partial A}{\partial t}\rangle + \frac{i}{\hbar}\langle[H, A]\rangle$

ANGULAR MOMENTUM

$[L_i, L_j] = i\hbar\epsilon ijk L_k$	$[L^2, L_i] = 0$	
$L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m}$	$L_z Y_{\ell m} = m\hbar Y_{\ell m}$	$-\ell \le m \le \ell$
$L_\pm = L_x \pm i L_y$	$L_\pm Y_{\ell m} = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\; Y_{\ell,m\pm 1}$	
$Y_{00} = \frac{1}{\sqrt{4\pi}}$	$Y_{11} = -\sqrt{\frac{3}{8\pi}}\; e^{i\phi}\; \sin\theta$	$Y_{10} = \sqrt{\frac{3}{4\pi}}\; \cos\theta$
$Y_{22} = \sqrt{\frac{15}{32\pi}}\; e^{2i\phi}\; \sin^2\theta$	$Y_{21} = -\sqrt{\frac{15}{8\pi}}\; e^{i\phi}\; \sin\theta\cos\theta$	$Y_{20} = \sqrt{\frac{5}{16\pi}}\; (3\cos^2\theta - 1)$
$Y_{\ell\ell} = e^{i\ell\phi}\; \sin^\ell\theta$	$Y_{\ell(-m)} = (-1)^\ell Y^*_{\ell m}$	$Y_{\ell m}(\pi-\theta, \phi+\pi) = (-1)^\ell\; Y_{\ell m}(\theta,\phi)$
$\frac{-\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] R_{n\ell}(r) + \left(V(r) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right) R_{n\ell}(r) = E R_{n\ell}(r)$		
$H = H_0 - \vec{\mu}\cdot\vec{B}$	$\vec{\mu} = \frac{-e}{2mc}\vec{L}$	$\vec{\mu} = \frac{-ge}{2mc}\vec{S}$
$S_i = \frac{\hbar}{2}\sigma_i$	$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$	$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$
$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$S_x = \hbar\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$	$S_y = \hbar\begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$	$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

HYDROGEN ATOM

$H = \frac{p^2}{2\mu} - \frac{Ze^2}{r}$	$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta,\phi)$	$E_n = -\frac{Z^2\alpha^2\mu c^2}{2n^2} = -\frac{13.6}{n^2}$ eV
$n = n_r + \ell + 1$	$n = 1, 2, 3, ...$	$\ell = 0, 1, ..., n-1$
$R_{n\ell}(\rho) = \rho^\ell \sum\limits_{k=0}^{\infty} a_k \rho^k e^{-\rho/2}$	$a_{k+1} = \frac{k+\ell+1-n}{(k+1)(k+2\ell+2)} a_k$	$\rho = \sqrt{\frac{-8\mu E}{\hbar^2}}\, r$
$R_{10} = 2(\frac{Z}{a_0})^{\frac{3}{2}}\; e^{\frac{-Zr}{a_0}}$	$R_{20} = 2(\frac{Z}{2a_0})^{\frac{3}{2}}(1 - \frac{Zr}{2a_0})\; e^{\frac{-Zr}{2a_0}}$	$R_{21} = \frac{1}{\sqrt{3}}(\frac{Z}{2a_0})^{\frac{3}{2}}(\frac{Zr}{a_0})\; e^{\frac{-Zr}{2a_0}}$
$R_{n,n-1} \;\propto\; r^{n-1}\; e^{-Zr/na_0}$	$\mu = \frac{m_1 m_2}{m_1+m_2}$	$\langle\psi_{n\ell m}\vert\frac{e^2}{r}\vert\psi_{n\ell m}\rangle = \frac{Ze^2}{n^2 a_0} = \frac{Z\alpha^2 mc^2}{n^2}$
$H_1 = -\frac{p^4}{8m^3c^2}$	$H_2 = \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L}$	$\Delta E_{12} = -\frac{1}{2n^3}\alpha^4 mc^2(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n})$
$H_3 = \frac{e^2 g_p}{3mM_pc^2}\vec{S}\cdot\vec{I}4\pi\delta^3(\vec{r})$	$\Delta E_3 = \frac{2g_p m\alpha^4 mc^2}{3M_p n^3}(f(f+1) - I(I+1) - \frac{3}{4})$	
$H_B = \frac{eB}{2mc}(L_z + 2S_z)$	$\Delta E_B = \frac{e\hbar B}{2mc}(1 \pm \frac{1}{2\ell+1})m_j$ for $j = \ell \pm \frac{1}{2}$	

ADDITION OF ANGULAR MOMENTUM

$\vec{J} = \vec{L} + \vec{S}$	$\vert\ell - s\vert \le j \le \ell + s$	$L\vec{\cdot}S = \frac{1}{2}(J^2 - L^2 - S^2)$
$\psi_{jm_j\ell s} = \sum\limits_{m_\ell m_s} C(jm_j;\ell m_\ell s m_s) Y_{\ell m_\ell}\chi_{sm_s} = \sum\limits_{m_\ell m_s}\langle jm_j\ell s\vert\ell m_\ell s m_s\rangle Y_{\ell m_\ell}\chi_{sm_s}$		
$\psi_{j,m_j} = \psi_{\ell+\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{\ell+m+1}{2\ell+1}} Y_{\ell m}\chi_+ + \sqrt{\frac{\ell-m}{2\ell+1}} Y_{\ell,m+1}\chi_-$		for $s = \frac{1}{2}$ and any ℓ
$\psi_{j,m_j} = \psi_{\ell-\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{\ell-m}{2\ell+1}} Y_{\ell m}\chi_+ - \sqrt{\frac{\ell+m+1}{2\ell+1}} Y_{\ell,m+1}\chi_-$		for $s = \frac{1}{2}$ and any ℓ

PERTURBATION THEORY AND RADIATIVE DECAYS

$E_n^{(1)} = \langle \phi_n \| H_1 \| \phi_n \rangle$	$E_n^{(2)} = \sum\limits_{k \neq n} \frac{\|\langle \phi_k \| H_1 \| \phi_n \rangle\|^2}{E_n^{(0)} - E_k^{(0)}}$	$c_{nk}^{(1)} = \frac{\langle \phi_k \| H_1 \| \phi_n \rangle}{E_n^{(0)} - E_k^{(0)}}$
$c_n(t) = \frac{1}{i\hbar} \int\limits_0^t dt' \; e^{i(E_n - E_i)t'/\hbar} \langle \phi_n \| V(t') \| \phi_i \rangle$		
Fermi's Golden Rule:	$\Gamma_{i \to f} = \frac{2\pi}{\hbar} \|\langle \psi_f \| V \| \psi_i \rangle\|^2 \rho_f(E)$	
$\Gamma_{i \to f} = \frac{2\pi}{\hbar} \int \prod\limits_k \left(\frac{V d^3 p_k}{(2\pi\hbar)^3}\right) \|M_{fi}\|^2 \; \delta^3(\text{momentum conservation}) \; \delta(\text{Energy conservation})$		
$\Gamma_{m \to k}^{rad} = \frac{\alpha}{2\pi m^2 c^2} \int d\Omega_p \omega_{km} \|\langle \phi_m \| e^{-i\vec{k}\cdot\vec{r}} \hat{\epsilon} \cdot \vec{p} \| \phi_k \rangle\|^2$		
$\Gamma_{m \to k}^{E1} = \frac{\alpha}{2\pi c^2} \int d\Omega_p \omega_{km}^3 \|\langle \phi_m \| \hat{\epsilon} \cdot \vec{r} \| \phi_k \rangle\|^2$		$\Delta l = \pm 1, \; \Delta s = 0$
$\hat{\epsilon} \cdot \hat{r} = \sqrt{\frac{4\pi}{3}} (\epsilon_z Y_{10} + \frac{-\epsilon_x + i\epsilon_y}{\sqrt{2}} Y_{11} + \frac{\epsilon_x + i\epsilon_y}{\sqrt{2}} Y_{1-1})$		$\hat{\epsilon} \cdot \vec{k} = 0$
$I(\omega) \propto \frac{\Gamma/2}{(\omega - \omega_0)^2 + (\Gamma/2)^2}$	$\Gamma_{collision} = P\sigma \sqrt{\frac{3}{mkT}}$	$(\frac{\Delta\omega}{\omega})_{Dopler} = \sqrt{\frac{kT}{mc^2}}$

ATOMS AND MOLECULES

Hund: 1) max s	2)max ℓ (allowed)	3) min j ($\leq \frac{1}{2}$ shell) else max j
$E_{rot} = \frac{\ell(\ell+1)\hbar^2}{2I} \approx \frac{1}{2000}$ eV	$E_{vib} = (n + \frac{1}{2})\hbar\omega \approx \frac{1}{50}$ eV	