Final Exam **Dec. 3, 2001** **Physics 130B**

Name____________________________

1. A hydrogen atom is in an eigenstate (ψ) of J^2, L^2, and of J_z such that $J^2\psi = \frac{15}{4}\hbar^2\psi$, $L^2\psi = 6\hbar^2\psi$, $J_z\psi = -\frac{1}{2}\hbar\psi$, and of course the electron's spin is $\frac{1}{2}$. Determine the quantum numbers of this state as well as you can. If a measurement of L_z is made, what are the possible outcomes and what are the probabilities of each. *(10 points)*
   Solution:

$$\begin{aligned}
J^2\psi &= \frac{15}{4}\hbar^2\psi = j(j+1)\hbar^2\psi = \frac{3}{2}\frac{5}{2}\hbar^2\psi \\
L^2\psi &= 6\hbar^2\psi = \ell(\ell+1)\hbar^2\psi = 2(3)\hbar^2\psi \\
J_z\psi &= -\frac{1}{2}\hbar\psi = m_j\hbar\psi \\
j &= \frac{3}{2} \\
\ell &= 2 \\
m_j &= -\frac{1}{2} \\
s &= \frac{1}{2}
\end{aligned}$$

We are in a state of definite total angular momentum j and z component of total angular momentum m_j (as well as ℓ and s of course). Now we measure L_z getting a value of $m_\ell\hbar$. We are not in an eigenstate of that operator so we must rewrite our state in terms of the eigenstates of L_z. Note that $j = \ell - \frac{1}{2}$ so we pick the right formula for adding spin one-half to angular momenetum ℓ.

$$\psi_{j,m_j} = \psi_{\ell-\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{\ell-m}{2\ell+1}}Y_{\ell m}\chi_+ - \sqrt{\frac{\ell+m+1}{2\ell+1}}Y_{\ell,m+1}\chi_-$$

Note also that $m_j = m_\ell + m_s$ so the dummy variable used in the forumla is $m = -1$.

$$\psi_{\frac{3}{2}\frac{-1}{2}} = \sqrt{\frac{3}{5}}Y_{2,-1}\chi_+ - \sqrt{\frac{2}{5}}Y_{2,0}\chi_-$$

The two possible measurements of L_z can be read from the spherical harmonic index (or just computed from m_j and m_s)
$-\hbar$ with a probability of $\frac{3}{5}$ and
0 with a probability of $\frac{2}{5}$.

2. An electron is in a three dimensional harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$. A small electric field, of strength E_z, is applied in the z direction. Calculate the lowest order nonzero correction to the ground state energy. *(10 points)*
   Solution:
   Working in Cartesian coordinates the ground state had all three of the oscillators in the n=0 state. The perturbation is $V = eEz = eE\sqrt{\frac{\hbar}{2m\omega}}(A_z + A_z^\dagger)$. First order perturbation theory gives zero energy shift since

$$\langle 000|A_z + A_z^\dagger|000\rangle = 0.$$

   In second order, we have

$$E_{000}^{(2)} = \sum_{n_x n_y n_z \neq 000} \frac{|\langle n_x n_y n_z|V|000\rangle|^2}{E_{000}^{(0)} - E_{n_x n_y n_z}^{(0)}} = e^2E^2\frac{\hbar}{2m\omega}\frac{|\langle 001|A_z^\dagger|000\rangle|^2}{\frac{3}{2}\hbar\omega - \frac{5}{2}\hbar\omega} = \frac{-e^2E^2}{2m\omega^2}$$

3. Calculate the splitting of the $n = 3$ states of Hydrogen in a weak magnetic field B. Draw a diagram showing the states before and after the field is applied *(10 points)*
   Solution:

$$\Delta E_{12} = -\frac{1}{2n^3}\alpha^4 mc^2\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) = -\frac{1}{54}\alpha^4 mc^2\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{12}\right)$$

   Before the field is applied the states split only according to the value of j, with

$$\Delta E_{\frac{1}{2}} = -\frac{\alpha^4 mc^2}{54}\frac{3}{4}$$
$$\Delta E_{\frac{3}{2}} = -\frac{\alpha^4 mc^2}{54}\frac{1}{4}$$
$$\Delta E_{\frac{5}{2}} = -\frac{\alpha^4 mc^2}{54}\frac{1}{12}$$

   With a weak applied B field

$$\Delta E_B = \frac{e\hbar B}{2mc}\left(1 \pm \frac{1}{2\ell+1}\right)m_j = \mu_B B g_L m_j$$

   with the gyromagnetic ratio g_L different for each state.

$$\begin{aligned} g_L &= \left(1 \pm \frac{1}{2\ell+1}\right) \\ &= 2 \qquad 3S_{\frac{1}{2}} \\ &= \frac{2}{3} \qquad 3P_{\frac{1}{2}} \\ &= \frac{4}{3} \qquad 3P_{\frac{3}{2}} \\ &= \frac{4}{5} \qquad 3D_{\frac{3}{2}} \\ &= \frac{6}{5} \qquad 3D_{\frac{5}{2}} \end{aligned}$$

   Now insert a nice diagram showing these sizes and the correct number of lines for each of the 4 states above which are split by the B field.

4. A hydrogen atom in the ground state is put in a magnetic field. Assume that the energy shift due to the B field is of the same order as the hyperfine splitting of the ground state. Find the eigenenergies of the (four) ground states as a function of the B field strength. Make sure you define any constants (like $\mathcal{A}$) you use in terms of fundamental constants. *(10 points)*

Solution:

Using the eigenstates of total spin $|fm_f\rangle$, the hyperfine perturbation is diagonal.

$$\Delta E_{hf} = \frac{2g_p m\alpha^4 mc^2}{3M_p n^3}(f(f+1) - \frac{3}{2}) \equiv C(f(f+1) - \frac{3}{2})$$

The energy shift in the B field is $2\mu_B B m_s$. The matrix for $\langle fm_f|H_{hf} + H_B|f'm'_f\rangle$ is

$$\begin{matrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 0 \end{matrix} \begin{pmatrix} \frac{C}{2} + \mu_B B & 0 & 0 & 0 \\ 0 & \frac{C}{2} - \mu_B B & 0 & 0 \\ 0 & 0 & \frac{C}{2} & \mu_B B \\ 0 & 0 & \mu_B B & \frac{-3C}{2} \end{pmatrix}.$$

The top part is already diagonal so we only need to work in bottom right 2 by 2 matrix, solving the eigenvalue problem.

$$\begin{pmatrix} x & y \\ y & -3x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = E\begin{pmatrix} a \\ b \end{pmatrix} \qquad where \qquad \begin{matrix} x \equiv \frac{C}{2} \\ y \equiv \mu_B B \end{matrix}$$

Setting the determinant equal to zero, we get

$$(x - E)(-3x - E) - y^2 = 0.$$

$$E^2 + 2xE - 3A^2 - y^2 = 0$$

$$E = \frac{-2x \pm \sqrt{4x^2 + 4(3x^2 + y^2)}}{2} = -x \pm \sqrt{x^2 + (3x^2 + y^2)}$$

$$= -x \pm \sqrt{4x^2 + y^2}$$

The eigenvalues for the $m_f = 0$ states, which mix differently as a function of the field strength, are

$$E = -\frac{C}{2} \pm \sqrt{C^2 + (\mu_B B)^2}.$$

The eigenvalues for the other two states which remain eigenstates independent of the field strength are

$$\frac{C}{2} + \mu_B B$$

and

$$\frac{C}{2} - \mu_B B.$$

5. Write out the Hamiltonian for Helium, ignoring fine structure and hyperfine terms. In our study of Helium, we use the electron wavefunctions that we derived for Hydrogen. Use these wavefunctions to write out the $m = 0$ substate of the ground, first excited, and second excited states of Helium. These states should explicitly have the correct symmetry to satisfy the Pauli principle. *(8 points)*

   Solution:

$$H = \frac{p_1^2}{2m} - \frac{2e^2}{r_1} + \frac{p_2^2}{2m} - \frac{2e^2}{r_2} + \frac{e^2}{|\vec{r_1} - \vec{r_2}|}$$

   The ground state of Helium is $(1s)^2$ and there is only one way to make it. The first and second excited states of Helium are both $(1s)(2s)$ and differ due to the symmetry of the spatial wavefunction, the antisymmetric having lower energy. We need to pick the state with $m = m_s = 0$ since all the electrons are in $\ell = 0$ states.

$$\psi_{gs} = \psi_{100}(\vec{r_1})\psi_{100}(\vec{r_2})\frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)})$$

   The space state is symmetric so the spin state must be anti.

$$\psi_{1st} = \frac{1}{\sqrt{2}}(\psi_{100}(\vec{r_1})\psi_{200}(\vec{r_2}) - \psi_{200}(\vec{r_1})\psi_{100}(\vec{r_2}))\frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)})$$

$$\psi_{2nd} = \frac{1}{\sqrt{2}}(\psi_{100}(\vec{r_1})\psi_{200}(\vec{r_2}) + \psi_{200}(\vec{r_1})\psi_{100}(\vec{r_2}))\frac{1}{\sqrt{2}}(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)})$$

6. Write down the ground state (in spectroscopic notation) for the element Oxygen ($Z = 8$). *(5 points)*

   Solution:

   Oxygen has 2 holes in the 2p shell: max s=1, ℓ must be 1 due to Pauli, and we couple to max j=2 since the shell is more than half full.

$$^3P_2$$

7. A Florine (Z=9) atom is in its ground state. Write the state in spectroscopic notation. (Ignore the spin of the nucleus.) A weak magnetic field is applied splitting the state. How many energy levels are there now? What is the gyromagnetic ratio for the ground state of Florine. *(8 points)*

   Solution:

   Florine has one hole in the 2p shell so there aren't any options except for j: $s = \frac{1}{2}$, $\ell = 1$, max $j = \frac{3}{2}$.

$$^2P_{\frac{3}{2}}$$

   Since $j = \frac{3}{2}$, there are 4 energy levels. We know how to compute the Lande g factor for a single electron. Its the same for a single hole.

$$g_L = \left(1 \pm \frac{1}{2\ell + 1}\right) = \left(1 + \frac{1}{2\ell + 1}\right) = \frac{4}{3}$$

8. Assume we have an electron in a standard one dimensional harmonic oscillator of frequency ω in its ground state. An weak electric field is applied for a time interval T. Calculate the probability to make a transition to the first excited state. *(10 points)*
Solution:

$$
\begin{aligned}
V &= eEx = eE\sqrt{\frac{\hbar}{2m\omega}}(A + A^\dagger) \\
b_n(t) &= \frac{1}{i\hbar}\int_0^t dt'\ e^{i(E_n - E_i)t'/\hbar}\langle\phi_n|V(t')|\phi_i\rangle \\
b_1(T) &= \frac{1}{i\hbar}\int_0^T dt'\ e^{i(E_1 - E_0)t'/\hbar} eE\sqrt{\frac{\hbar}{2m\omega}}\langle 1|A + A^\dagger|0\rangle \\
&= \frac{1}{i\hbar} eE\sqrt{\frac{\hbar}{2m\omega}}\int_0^T dt'\ e^{i\omega t'}\langle 1|A^\dagger|0\rangle \\
&= \frac{1}{i\hbar} eE\sqrt{\frac{\hbar}{2m\omega}}\left[\frac{e^{i\omega t'}}{i\omega}\right]_0^T \\
&= -eE\sqrt{\frac{1}{2m\hbar\omega^3}}\left[e^{i\omega T} - 1\right] \\
&= -eE\sqrt{\frac{1}{2m\hbar\omega^3}}e^{i\omega T/2}\left[e^{i\omega T/2} - e^{-i\omega T/2}\right] \\
&= -eE\sqrt{\frac{1}{2m\hbar\omega^3}}e^{i\omega T/2} 2i\sin\left(\frac{\omega T}{2}\right) \\
&= -ieE\sqrt{\frac{2}{m\hbar\omega^3}}e^{i\omega T/2}\sin\left(\frac{\omega T}{2}\right) \\
P_1 &= \frac{2e^2E^2}{m\hbar\omega^3}\sin^2\left(\frac{\omega T}{2}\right)
\end{aligned}
$$