Midterm **Nov. 10, 2008** **Physics 130B** **Secret No.=**

1. At $t = 0$, an electron is in an eigenstate of S_y with eigenvalue $-\frac{\hbar}{2}$. A magnetic field B is applied in the z direction.

a) What is the state at $t = 0$ (in the usual S_z basis)?

$$\chi(t=0) = \chi_-^{(y)}$$
$$\sigma_y \chi_-^{(y)} = -\chi_-^{(y)}$$
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ ia \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$
$$a = ib$$
$$\chi(t=0) = \chi_-^{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

b) What is the state at a later time t?

$$H_B = -\vec{\mu} \cdot \vec{B} = \frac{eg}{2mc} B S_z$$
$$E = \pm \mu_B B$$
$$\chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{-i\mu_b Bt/\hbar} \\ e^{i\mu_b Bt/\hbar} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}$$

c) Find the expectation value of S_x as a function of time.

$$\langle \chi(t) | S_x | \chi(t) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -ie^{i\omega t} & e^{-i\omega t} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{-i\omega t} \\ e^{i\omega t} \end{pmatrix} = \frac{\hbar}{4} \left(-ie^{2i\omega t} + ie^{-2i\omega t} \right) = \frac{\hbar}{2} \sin(2\omega t)$$

d) What is the probability to measure that the electron's spin along the x direction is $\frac{\hbar}{2}$ as a function of time?

$$P_+^{(x)} = \left| \langle \chi_+^{(x)} | \chi(t) \rangle \right|^2$$
$$\sigma_x \chi_+^{(x)} = \chi_+^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$P_+^{(x)} = \frac{1}{4} \left| \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} ie^{-i\omega t} \\ e^{i\omega t} \end{pmatrix} \right|^2 = \frac{1}{4} \left| ie^{-i\omega t} + ie^{i\omega t} \right|^2$$
$$= \frac{1}{4} (1 + 1 + ie^{-2i\omega t} - ie^{2i\omega t}) = \frac{1}{2} (1 + \sin(2\omega t))$$

2. Consider a two dimensional harmonic oscillator problem described by the Hamiltonian

$$H_0 = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2).$$

a) Accurately calculate the energy shifts to the degenerate first excited states, to first order, if the additional weak potential $H_1 = \gamma xy$ is applied. (Use the solution in Cartesian coordinates.)

$$u_0^{(x)} u_1^{(y)} = |01\rangle \qquad u_1^{(x)} u_0^{(y)} = |10\rangle$$

$$H_1 = \gamma xy = \frac{\gamma \hbar}{2m\omega}(A_x + A_x^\dagger)(A_y + A_y^\dagger)$$

$$\langle 01|H_1|01\rangle = \langle 10|H_1|10\rangle = 0 \qquad \langle 01|H_1|10\rangle = \langle 10|H_1|01\rangle = \frac{\gamma\hbar}{2m\omega}$$

$$\frac{\gamma\hbar}{2m\omega}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = E\begin{pmatrix} a \\ b \end{pmatrix} \qquad E = \pm\frac{\gamma\hbar}{2m\omega}$$

b) What are the new energy eigenstates for the degenerate first excited state (in terms of the standard eigenstates $u_0^{(x)} u_1^{(y)}$ and $u_1^{(x)} u_0^{(y)}$)?

$$\psi_\pm = \frac{1}{\sqrt{2}}(u_0^{(x)} u_1^{(y)} \pm u_1^{(x)} u_0^{(y)})$$

3. **a)** Calculate the fine structure energy shifts (in eV!) for the $n = 4$ states of Hydrogen. Include the effects of relativistic corrections, the spin-orbit interaction, and the so-called Darwin term (due to Dirac equation). Do not include hyperfine splitting. Clearly list the states in spectroscopic notation and give the energy shift for each.

$$\Delta E_{12} = -\frac{1}{2n^3}\alpha^4 mc^2 (\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}) = -\frac{1}{64}\alpha^2(\frac{1}{2}\alpha^2 mc^2)(\frac{1}{j+\frac{1}{2}} - \frac{3}{16})$$

$$= -\frac{1}{64}\left(\frac{1}{137}\right)^2 (13.6 \quad eV)(\frac{1}{j+\frac{1}{2}} - \frac{3}{16}) = -11.3 \quad \mu eV(\frac{1}{j+\frac{1}{2}} - \frac{3}{16})$$

The states are $4S_{\frac{1}{2}}$, $4P_{\frac{1}{2}}$, $4P_{\frac{3}{2}}$, and $4D_{\frac{3}{2}}$, $4D_{\frac{5}{2}}$, $4F_{\frac{5}{2}}$, $4F_{\frac{7}{2}}$, with factors of $(\frac{1}{j+\frac{1}{2}} - \frac{3}{16}) = \frac{13}{16}, \frac{13}{16}, \frac{5}{16}, \frac{5}{16}, \frac{7}{48}, \frac{7}{48}$, and $\frac{1}{16}$ respectively. This gives shifts 0f -9.2, -9.2, -3.5, -3.5, -1.65 -1.65, and -0.71 μeV.
b) Calculate and show the splitting of the $n = 4$ states (from part a) in a weak magnetic field B. Draw a diagram showing the states before and after the field is applied.

$$\Delta E_B = \frac{e\hbar B}{2mc}(1 \pm \frac{1}{2\ell+1})m_j = \mu_B B g_L m_j$$

The states are $4S_{\frac{1}{2}}$, $4P_{\frac{1}{2}}$, $4P_{\frac{3}{2}}$, and $4D_{\frac{3}{2}}$, $4D_{\frac{5}{2}}$, $4F_{\frac{5}{2}}$, $4F_{\frac{7}{2}}$, with $g_L = 1 \pm \frac{1}{2\ell+1} = 2, \frac{2}{3}, \frac{4}{3}, \frac{4}{5}, \frac{6}{5}, \frac{6}{7}$, and $\frac{8}{7}$, respectively.