Physics 130B Final Exam Dec. 8, 2008

1. A hydrogen atom is in a uniform electric field in the z direction which turns on abruptly at $t = 0$ and decays exponentially as a function of time, $E(t) = E_0 e^{-\gamma t}$. The atom is initially in its ground state. Find the probability for the atom to have made a transition to the $2P$ state as $t \to \infty$. What z components of orbital angular momentum are allowed in the $2P$ states generated by this transition? *(8 points)*

   We use time dependent perturbation theory with the purturbation $V = eE_0 e^{-\gamma t} z$.

$$c_n(\infty) = \frac{1}{i\hbar}\int\limits_0^\infty dt' e^{i(E_n - E_i)t'/\hbar} \langle \phi_{21m}|V(t')|\phi_{100}\rangle$$

$$c_n(\infty) = \frac{eE_0}{i\hbar}\int\limits_0^\infty dt' e^{(i\omega_{ni} - \gamma)t'} \langle \phi_{21m}|z|\phi_{100}\rangle$$

$$c_n(\infty) = \frac{eE_0}{i\hbar}\frac{1}{i\omega_{ni} - \gamma}\sqrt{\frac{4\pi}{3}} \langle \phi_{21m}|rY_{10}|\phi_{100}\rangle$$

$$c_n(\infty) = \frac{eE_0}{i\hbar}\frac{1}{i\omega_{ni} - \gamma}\sqrt{\frac{4\pi}{3}}\sqrt{\frac{1}{4\pi}}\delta_{m0}\int\limits_0^\infty r^3 R_{21}^* R_{10} dr$$

$$c_n(\infty) = \frac{eE_0}{\sqrt{3}i\hbar}\frac{1}{i\omega_{ni} - \gamma}\delta_{m0}\int\limits_0^\infty r^3 \sqrt{\frac{1}{3}}\left(\frac{1}{2a_0}\right)^{\frac{3}{2}}\left(\frac{r}{a_0}\right) e^{-r/2a_0} 2\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-r/a_0} dr$$

$$c_n(\infty) = \frac{eE_0}{3i\hbar}\frac{1}{i\omega_{ni} - \gamma}\delta_{m0}\frac{1}{\sqrt{2}a_0^4}\int\limits_0^\infty r^4 e^{-3r/2a_0} dr$$

$$c_n(\infty) = \frac{eE_0}{3i\hbar}\frac{1}{i\omega_{ni} - \gamma}\delta_{m0}\frac{1}{\sqrt{2}a_0^4}\frac{4!2^5 a_0^5}{3^5}$$

$$c_n(\infty) = \frac{eE_0}{i\hbar}\frac{1}{i\omega_{ni} - \gamma}\delta_{m0}\frac{2^7\sqrt{2}}{3^5}a_0$$

$$P_n(\infty) = \frac{2^{15}}{3^{10}}\frac{e^2 E_0^2 a_0^2}{\hbar^2}\frac{1}{\omega_{ni}^2 + \gamma^2}\delta_{m0}$$

The only 2P state excited will be the $m = 0$ state.

2. The u axis is in the xy plane just in between the two axes ($\phi = \frac{\pi}{4}$). So the spin operator for the u axis is given by $S_u = \frac{1}{\sqrt{2}}(S_x + S_y)$. Find the eigenvectors of S_u for a spin $\frac{1}{2}$ particle in the usual basis. *(5 points)*

The eigenvalues of S_u will be $\pm\frac{\hbar}{2}$ as for any spin $\frac{1}{2}$ component.

$$S_u = \frac{\hbar}{2}\begin{pmatrix} 0 & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \frac{1-i}{\sqrt{2}}b \\ \frac{1+i}{\sqrt{2}}a \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\chi_+^{(u)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1+i}{2} \end{pmatrix}$$

$$\chi_-^{(u)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1+i}{2} \end{pmatrix}$$

3. The lowest energy excited states of Helium have the electrons in a (1s)(2s) configuration in space and can have total spin 0 or 1. Write out the full (space and spin) states for **a)** $s = 1$ and $m_s = 0$ and for **b)** $s = 0$ and $m_s = 0$. Make sure to take account of the Pauli principle in your states. Which state will have the lower energy? *(6 points)*

a) $s = 1$ and $m_s = 0$

$$\psi = \frac{1}{2}\left(R_{10}(r_1)R_{20}(r_2) - R_{20}(r_1)R_{10}(r_2)\right)Y_{00}Y_{00}\left(\chi_+^{(1)}\chi_-^{(2)} + \chi_-^{(1)}\chi_+^{(2)}\right)$$

b) $s = 0$ and $m_s = 0$

$$\psi = \frac{1}{2}\left(R_{10}(r_1)R_{20}(r_2) + R_{20}(r_1)R_{10}(r_2)\right)Y_{00}Y_{00}\left(\chi_+^{(1)}\chi_-^{(2)} - \chi_-^{(1)}\chi_+^{(2)}\right)$$

The $s = 1$ state has lower energy because the spatial state is antisymmetric, reducing the repulsion between the electrons.

4. We want to find the eigenstates of total J^2 and J_z for a spin $\frac{3}{2}$ particle in an $\ell = 1$ spatial state ($\vec{J} = \vec{L} + \vec{S}$). **a)** What are the allowed values of j, the total angular momentum quantum number? **b)** Write down the state of maximum m_j for the maximum j state. Use $|jm_j\rangle$ notation and $|\ell m_\ell\rangle|sm_s\rangle$ for the product states. **c)** Now apply the lowering operator to get the other m_j states. You only need to go down to $m_j = \frac{1}{2}$ because of the obvious symmetry. **d)** Now find the states with the next highest value of j in a similar way. *(10 points)*

a) The possible values of j are $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$.

b)

$$|jm_j\rangle = |\ell m_\ell\rangle\,|sm_s\rangle$$

$$\left|\frac{5}{2}\frac{5}{2}\right\rangle = |11\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle$$

c)

$$J_-\left|\frac{5}{2}\frac{5}{2}\right\rangle = (L_- + S_-)\,|11\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle$$

$$\sqrt{\frac{5}{2}\frac{7}{2} - \frac{5}{2}\frac{3}{2}}\left|\frac{5}{2}\frac{3}{2}\right\rangle = \sqrt{2}\,|10\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + \sqrt{\frac{3}{2}\frac{3}{2} - \frac{3}{2}\frac{1}{2}}\,|11\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle$$

$$\left|\frac{5}{2}\frac{3}{2}\right\rangle = \sqrt{\frac{2}{5}}\,|10\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + \sqrt{\frac{3}{5}}\,|11\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle$$

$$\sqrt{\frac{5}{2}\frac{7}{2} - \frac{3}{2}\frac{1}{2}}\left|\frac{5}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\sqrt{2}\,|1-1\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + \sqrt{\frac{2}{5}}\sqrt{\frac{3}{2}\frac{5}{2} - \frac{3}{2}\frac{1}{2}}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\sqrt{2}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}}\sqrt{\frac{3}{2}\frac{5}{2} + \frac{1}{2}\frac{1}{2}}\,|11\rangle\left|\frac{3}{2} - \frac{1}{2}\right\rangle$$

$$\sqrt{8}\left|\frac{5}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{4}{5}}\,|1-1\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + 2\sqrt{\frac{6}{5}}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle + 2\sqrt{\frac{3}{5}}\,|11\rangle\left|\frac{3}{2} - \frac{1}{2}\right\rangle$$

$$\left|\frac{5}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{1}{10}}\,|1-1\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + \sqrt{\frac{6}{10}}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{3}{10}}\,|11\rangle\left|\frac{3}{2} - \frac{1}{2}\right\rangle$$

d)

$$\left|\frac{3}{2}\frac{3}{2}\right\rangle = \sqrt{\frac{3}{5}}\,|10\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle - \sqrt{\frac{2}{5}}\,|11\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle$$

$$\sqrt{\frac{15}{4} - \frac{3}{4}}\left|\frac{3}{2}\frac{1}{2}\right\rangle = \sqrt{2}\sqrt{\frac{3}{5}}\,|1-1\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + \sqrt{\frac{15}{4} - \frac{3}{4}}\sqrt{\frac{3}{5}}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle - \sqrt{2}\sqrt{\frac{2}{5}}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle - \sqrt{\frac{15}{4} + \frac{1}{4}}\sqrt{\frac{2}{5}}\,|11\rangle\left|\frac{3}{2} - \frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\,|1-1\rangle\left|\frac{3}{2}\frac{3}{2}\right\rangle + \sqrt{\frac{1}{15}}\,|10\rangle\left|\frac{3}{2}\frac{1}{2}\right\rangle - \sqrt{\frac{8}{15}}\,|11\rangle\left|\frac{3}{2} - \frac{1}{2}\right\rangle$$

5. A hydrogen atom is in the $3D_{\frac{3}{2}}$ energy eigenstate with $m_j = +\frac{1}{2}$: ($n = 3$, $j = \frac{3}{2}$, $m_j = +\frac{1}{2}$, $\ell = 2$). A measurement of the z component of orbital angular momentum (L_z) is made. What are the possible outcomes of this measurement and what are the probabilities of each? You may ignore nuclear spin in this problem. *(5 points)*

This state has $j = \ell - \frac{1}{2}$ so we will use the formula for that.

$$\psi_{j,m_j} = \psi_{\ell-\frac{1}{2},m+\frac{1}{2}} = \sqrt{\frac{\ell - m}{2\ell + 1}} Y_{\ell m}\chi_+ - \sqrt{\frac{\ell + m + 1}{2\ell + 1}} Y_{\ell,m+1}\chi_-$$

Using this formula for the $D_{\frac{3}{2}}$ state we have $\ell = 2$ and $m = 0$.

$$\psi_{\frac{3}{2},\frac{1}{2}} = \sqrt{\frac{2}{5}} Y_{20}\chi_+ - \sqrt{\frac{3}{5}} Y_{21}\chi_-$$

If we measure m we get $m = 0$ with a probability of $\frac{2}{5}$ or $m = 1$ with a probability of $\frac{3}{5}$.

6. Calculate the energies (relative to the H ground state energy) of the four hyperfine ground states of hydrogen in an intermediate strength magnetic field. (The field is strong enough so that the perturbation is similar in size to the hyperfine splitting.) *(10 points)*

The hyperfine splitting is given by:

$$\Delta E_3 = \frac{2g_p m\alpha^4 mc^2}{3M_p n^3}(f(f+1) - I(I+1) - \frac{3}{4}) = \frac{\mathcal{A}}{2}\left(f(f+1) - \frac{3}{2}\right)$$

This gives hyperfine energy shifts of $-\frac{3\mathcal{A}}{4}$ for $f = 0$ and $+\frac{\mathcal{A}}{4}$ for $f = 1$. The perturbation due the field acts on the electron's magnetic moment, $V = -\vec{\mu}_e \cdot \vec{B} = \mu_B B\sigma_z$. We will work in the basis states $|fm_f\rangle$ and evaluate the matrix of the perturbation by writing those states in terms of the $|m_e m_p\rangle$ states.

$$\begin{aligned}
\langle 11|\sigma_z|11\rangle &= \langle ++|\sigma_z|++\rangle = 1 \\
\langle 11|\sigma_z|1-1\rangle &= \langle ++|\sigma_z|--\rangle = 0 \\
\langle 1-1|\sigma_z|1-1\rangle &= \langle --|\sigma_z|--\rangle = -1 \\
\langle 11|\sigma_z|10\rangle &= \langle ++|\sigma_z\frac{1}{\sqrt{2}}(|+-\rangle + |--\rangle) = 0 \\
\langle 11|\sigma_z|00\rangle &= \langle 1-1|\sigma_z|00\rangle = \langle 1-1|\sigma_z|10\rangle = 0 \\
\langle 00|\sigma_z|10\rangle &= \langle\frac{1}{\sqrt{2}}(\langle +-| - \langle -+|)|\sigma_z\frac{1}{\sqrt{2}}(|+-\rangle + |--\rangle) = 1 \\
H &= \begin{pmatrix} -\frac{3\mathcal{A}}{4} & \mu_B B & 0 & 0 \\ \mu_B B & \frac{\mathcal{A}}{4} & 0 & 0 \\ 0 & 0 & \frac{\mathcal{A}}{4} + \mu_B B & 0 \\ 0 & 0 & 0 & \frac{\mathcal{A}}{4} - \mu_B B \end{pmatrix}
\end{aligned}$$

Two of the energies are clear in the block diagonal Hamiltonian matrix. We need to compute the other two by diagonalizing the upper left part of the matrix.

$$\begin{vmatrix} -\frac{3\mathcal{A}}{4} - \lambda & \mu_B B \\ \mu_B B & \frac{\mathcal{A}}{4} - \lambda \end{vmatrix} = 0$$

$$\lambda = -\frac{\mathcal{A}}{4} \pm \sqrt{\left(\frac{\mathcal{A}}{2}\right)^2 + (\mu_B B)^2}$$

So the four energies are $\frac{\mathcal{A}}{4} \pm \mu_B B$ and $-\frac{\mathcal{A}}{4} \pm \sqrt{\left(\frac{\mathcal{A}}{2}\right)^2 + (\mu_B B)^2}$. They give the right value for $B = 0$.

7. **a)** Write down the ground state (in spectroscopic notation) for the element Sulphur ($Z = 16$).
The electron configuration is $(1s)^2(2s)^2(2p)^6(3s)^2(3p)^4$, so its two holes in the 3P shell. The first Hund's rule says max s so $s = 1$. This is symmetric in spin so we need antisymmetric in ℓ. The antisymmetric state is $\ell = 1$. Since the shell is more than half full we go for maximum j which is $\ell + s = 2$. So the state is ${}^3P_2$.
b) Write down the ground state (in spectroscopic notation) for the element Aluminum ($Z = 13$).
This is one 3P electron outside a closed shell so the ground state is ${}^2P_{\frac{1}{2}}$.
c) If we apply a magnetic field, what is the Lande g factor for this ground state of Aluminum.
With one electron, the Lande g factor works the same way as for Hydrogen so $g_L = 1 \pm \frac{1}{2\ell+1}$. Since this is $j = \ell - \frac{1}{2}$, we use the minus sign and $\ell = 1$ giving $g_L = 1 - \frac{1}{3} = \frac{2}{3}$. *(6 points)*

8. Calculate the total decay rate for the $3s \longrightarrow 2p$ transition in hydrogen. What is the lifetime in nanoseconds? You may find the radial wave function $R_{30} = \frac{2}{3\sqrt{3}} \frac{1}{(a_0)^{\frac{3}{2}}} \left(1 - \frac{2r}{3a_0} + \frac{1}{6}\left(\frac{2r}{3a_0}\right)^2\right) e^{-r/3a_0}$ useful. *(8 points)*
We can use the formula that is averaged over initial states and summed over final polarizations, states, and photon direction. We have $\ell = 0$ and $\ell' = \ell + 1$. Note that getting the correct constants and units in the formula is more important than the number of order 1 that comes out of the integral. Nevertheless, we do the integral after making it dimensionless.

$$\Gamma = \frac{4\alpha\omega_{in}^3}{3c^2}\frac{1}{1}\left|\int_0^\infty R_{21}^* R_{30} r^3 dr\right|^2$$

$$\Gamma = \frac{4\alpha\omega_{in}^3}{3c^2}\left|\int_0^\infty \left(\frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{\frac{3}{2}}\frac{r}{a_0}e^{-r/2a_0}\right)\left(\frac{2}{3\sqrt{3}}\frac{1}{a_0^{\frac{3}{2}}}\left(1 - \frac{2r}{3a_0} + \frac{1}{6}\left(\frac{2r}{3a_0}\right)^2\right)e^{-r/3a_0}\right) r^3 dr\right|^2$$

$$\Gamma = \frac{2\alpha\omega_{in}^3 a_0^8}{3^5 c^2 a_0^6}\left|\int_0^\infty x e^{-x/2}\left(1 - \frac{2}{3}x + \frac{4}{54}x^2\right)e^{-x/3}x^3 dx\right|^2 = \frac{2\alpha\omega_{in}^3 a_0^2}{3^5 c^2}\left|\int_0^\infty \left(x^4 - \frac{2}{3}x^5 + \frac{2}{27}x^6\right)e^{-5x/6}dx\right|^2$$

$$\Gamma = \frac{2\alpha\omega_{in}^3 a_0^2}{3^5 c^2}\left|\left(24\left(\frac{6}{5}\right)^5 - \frac{2}{3}120\left(\frac{6}{5}\right)^6 + \frac{2}{27}720\left(\frac{6}{5}\right)^7\right)\right|^2 = \frac{2\alpha\omega_{in}^3 a_0^2}{3^5 c^2}\left(\frac{6}{5}\right)^{10}\left|24 - 80\frac{6}{5} + \frac{2}{3}80\left(\frac{6}{5}\right)^2\right|^2$$

$$\Gamma = \frac{\alpha(\hbar\omega_{in})^3 a_0^2 c}{(\hbar c)^3}\frac{2^{11}3^5}{5^{10}}\left|24 - 96 + 64\left(\frac{6}{5}\right)\right|^2 = \frac{2\alpha\omega_{in}^3 a_0^2}{3^5 c^2}\left(\frac{6}{5}\right)^{10}\left|24 - 80\frac{6}{5} + \frac{2}{3}80\left(\frac{6}{5}\right)^2\right|^2$$

$$\Gamma = \frac{\alpha(\hbar\omega_{in})^3 a_0^2 c}{(\hbar c)^3}\frac{2^{11}3^5}{5^{10}}\left|24 - 96 + 96\left(\frac{4}{5}\right)\right|^2 = \frac{\alpha(\hbar\omega_{in})^3 a_0^2 c}{(\hbar c)^3}\frac{2^{11}3^5}{5^{10}}\left|24 - \frac{96}{5}\right|^2$$

$$\Gamma = 1.17\frac{\alpha(\hbar\omega_{in})^3 a_0^2 c}{(\hbar c)^3}$$

$$\Gamma = 1.17\frac{\left(\frac{5}{36}13.6eV\right)^3(0.53\mathring{A})^2(3 \times 10^{18}\mathring{A}/s)}{137(1973eV\mathring{A})^3} = 6.3 \times 10^6/s$$

$$\tau = 160ns$$

9. What perturbation causes the decay of the 2P state of Hydrogen to the 1S state if no external field is applied? Qualitatively, what step did we make in our calculation of stimulated emission in harmonic fields to allow for emission with no applied field? *(5 points)*
We accounted for the zero point energy of the Harmonic Oscillator that is the EM field at (any) frequency ω by replacing the $\sqrt{N}$ with a $\sqrt{N+1}$ in the emission (raising) term of the radiation perturbation. Then, for no applied field ($N = 0$), we just have at 1 and there is still a perturbation that goes like $\hat{\epsilon} \cdot \vec{p}_e$ that causes decay of excited states.

10. If I make a gauge transformation of the EM field like $\vec{A} \rightarrow \vec{A} - \vec{\nabla} f(\vec{r}, t)$ and $\phi \rightarrow \phi + \frac{1}{c}\frac{\partial f(\vec{r},t)}{\partial t}$, qualitatively what must I do to the electron's wavefunction so that the Schrödinger equation remains invariant? *(5 points)*
If we make a local gauge transformation, we need to ma a local phase transformation of the electrons wavefunction $\psi \rightarrow e^{-\lambda(\vec{r},t)}\psi$ where λ is proportional to f with constants that should be equal to $\frac{1}{e}$ if we had $\hbar = c = 1$.