1. Find the (normalized) eigenvectors and eigenvalues of the $S_x$ (matrix) operator for $s = 1$ in the usual ($S_z$) basis.

2. A spin 1 system is in the following state in the usual $L_z$ basis: $\chi = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{2} \\ 1 + i \\ -i \end{pmatrix}$. What is the probability that a measurement of the $x$ component of spin yields zero? What is the probability that a measurement of the $y$ component of spin yields $+\hbar$?

3. A spin $\frac{1}{2}$ system is in the following state in the usual $S_z$ basis: $\chi = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{3} \\ 1 + i \end{pmatrix}$. What is the probability that a measurement of the $x$ component of spin yields $+\frac{\hbar}{2}$?

4. At $t = 0$, an electron is the spin state

$$\psi(t = 0) = \begin{pmatrix} 2i \\ \sqrt{3} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$ 

A magnetic field $B$ is applied in the $z$ direction. a) Find the spin state of the particle as a function of time. b) Find the expectation value of $S_y$ as a function of time. c) What is the probability to measure that the electron’s spin along the $x$ direction is $\frac{\hbar}{2}$ as a function of time? (10 points)
5. A spin 1 particle is in an $\ell = 2$ state. **a)** Find the allowed values of the total angular momentum quantum number, $j$. **b)** Write out the $|j, m_j\rangle$ states for the largest allowed $j$ value, in terms of the $|\ell, m_{\ell}\rangle|s, m_s\rangle$ basis. (That is give one state for every $m_j$ value.) **c)** If the particle is prepared in the state $|\ell = 2, m_{\ell} = 0\rangle|s = 1, m_s = 0\rangle$, what is the probability to measure $J^2 = 2\hbar^2$?

**a)** This is a simple addition of angular momentum problem $\vec{J} = \vec{L} + \vec{S}$ but its not one for which we will know the answer. We have to do the calculation. The total angular momentum quantum number $j$ is in the range $|\ell - s| \leq j \leq \ell + s$ and takes integer steps between the limits. In this case $\ell = 2$ and $s = 1$ giving $1 \leq j \leq 3$ or $j = 1, 2, 3$.

**b)** In this part of the problem we are writing the total angular momentum states in which $j, m_j, \ell, s$ are good quantum numbers in terms of products of states for which $\ell, m_{\ell}, s, m_s$ are good quantum numbers. To do this we will first write down the only possible (product) state for the $j = 3$, $m_j = 3$ total angular momentum state. Remember that the $z$ components of the angular momentum add so that the only way to get $m_j = 3$ is to have $m_{\ell} = 2$ and $m_s = 1$.

$$|j = 3, m_j = 3\rangle = |\ell = 2, m_{\ell} = 2\rangle|s = 1, m_s = 1\rangle$$

We will now economize in notation a bit and rewrite the same equation.

$$|3, 3\rangle = |2, 2\rangle|1, 1\rangle$$

We now apply the lowering operator to get the other states for $j = 3$ which is all the problem asks for; $J_- = L_- + S_-$. The equations for the lowering operators are given in the formulas of course.

$$J_-|3, 3\rangle = (L_- + S_-)|2, 2\rangle|1, 1\rangle = L_-|2, 2\rangle|1, 1\rangle + S_-|2, 2\rangle|1, 1\rangle$$

$$h\sqrt{12 - 6}|3, 2\rangle = h\sqrt{6 - 2}|2, 1\rangle|1, 1\rangle + h\sqrt{2 - 0}|2, 2\rangle|1, 0\rangle$$

$$|3, 2\rangle = \sqrt{\frac{4}{6}}|2, 1\rangle|1, 1\rangle + \sqrt{\frac{2}{6}}|2, 2\rangle|1, 0\rangle$$

$$|3, 2\rangle = \sqrt{\frac{2}{3}}|2, 1\rangle|1, 1\rangle + \sqrt{\frac{1}{3}}|2, 2\rangle|1, 0\rangle$$

Its good see the the state is normalized to check for arithmetic mistakes. We now lower the $z$ component of angular momentum again. We will skip some of the simple steps shown above.

$$J_-|3, 2\rangle = (L_- + S_-)\left(\sqrt{\frac{2}{3}}|2, 1\rangle|1, 1\rangle + \sqrt{\frac{1}{3}}|2, 2\rangle|1, 0\rangle\right)$$

$$\sqrt{10}|3, 1\rangle = \sqrt{\frac{2}{3}}\sqrt{6}|2, 0\rangle|1, 1\rangle + \sqrt{\frac{2}{3}}\sqrt{2}|2, 1\rangle|1, 0\rangle + \sqrt{\frac{1}{3}}\sqrt{4}|2, 1\rangle|1, 0\rangle + \sqrt{\frac{1}{3}}\sqrt{2}|2, 2\rangle|1, -1\rangle$$

$$|3, 1\rangle = \sqrt{\frac{12}{30}}|2, 0\rangle|1, 1\rangle + 2\sqrt{\frac{4}{30}}|2, 1\rangle|1, 0\rangle + \frac{2}{30}|2, 2\rangle|1, -1\rangle$$

$$|3, 1\rangle = \sqrt{\frac{6}{15}}|2, 0\rangle|1, 1\rangle + \sqrt{\frac{8}{15}}|2, 1\rangle|1, 0\rangle + \frac{1}{15}|2, 2\rangle|1, -1\rangle$$

$$J_-|3, 1\rangle = (L_- + S_-)\left(\sqrt{\frac{6}{15}}|2, 0\rangle|1, 1\rangle + \sqrt{\frac{8}{15}}|2, 1\rangle|1, 0\rangle + \frac{1}{15}|2, 2\rangle|1, -1\rangle\right)$$
\[ \sqrt{12} |3, 0\rangle = \sqrt{\frac{6}{15}} \sqrt{6} |2, -1\rangle |1, 1\rangle + \sqrt{2} \sqrt{\frac{6}{15}} |2, 0\rangle |1, 0\rangle \\
+ \sqrt{\frac{8}{15}} \sqrt{6} |2, 0\rangle |1, 0\rangle + \sqrt{\frac{8}{15}} \sqrt{2} |2, 1\rangle |1, -1\rangle + \sqrt{\frac{1}{15}} \sqrt{4} |2, 1\rangle |1, -1\rangle \]

\[ |3, 0\rangle = \sqrt{\frac{36}{180}} |2, -1\rangle |1, 1\rangle + 3 \sqrt{\frac{12}{180}} |2, 0\rangle |1, 0\rangle + 3 \sqrt{\frac{4}{180}} |2, 1\rangle |1, -1\rangle \]

\[ |3, 0\rangle = \sqrt{\frac{36}{180}} |2, -1\rangle |1, 1\rangle + \sqrt{\frac{108}{180}} |2, 0\rangle |1, 0\rangle + \sqrt{\frac{36}{180}} |2, 1\rangle |1, -1\rangle \]

\[ |3, 0\rangle = \sqrt{\frac{6}{15}} |2, 0\rangle |1, -1\rangle + \sqrt{\frac{8}{15}} |2, -1\rangle |1, 0\rangle + \sqrt{\frac{1}{15}} |2, -2\rangle |1, 1\rangle \]

\[ |3, -2\rangle = \sqrt{\frac{2}{3}} |2, -1\rangle |1, -1\rangle + \sqrt{\frac{1}{3}} |2, -2\rangle |1, 0\rangle \]

\[ |3, -3\rangle = |2, -2\rangle |1, -1\rangle \]

c) Measuring \( J^2 = 12\hbar^2 \) means measuring that \( j = 3 \). Since the z components of angular momentum add, \( m_j = 0 \) is required. So we simply compute the probability given the state \( \chi = |\ell = 2, m_l = 0\rangle |s = 1, m_s = 0\rangle \). We can read this from the 3, 0 state we computed in part (b).

\[ P = |\langle j = 3, m_j = 0 | \chi \rangle|^2 = \frac{9}{15} \]
6. We want to find the eigenstates of total $S^2$ and $S_z$ for two spin 1 particles which have an $S_1 \cdot S_2$ interaction. ($\vec{S} = \vec{S}_1 + \vec{S}_2$)

(a) What are the allowed values of $s$, the total spin quantum number.

(b) Write down the state of maximum $m_s$ for the maximum $s$ state. Use $|s m_s\rangle$ notation and $|s_1 m_1\rangle|s_2 m_2\rangle$ for the product states.

(c) Now apply the lowering operator to get the other $m_s$ states. You only need to go down to $m_s = 0$ because of the obvious symmetry.

(d) Now find the states with the other values of $s$ in a similar way.

This is very similar to the previous problem. We are adding angular momentum $\vec{S} = \vec{S}_1 + \vec{S}_2$. The allowed values of $s$ are given by $|s_1 - s_2| \leq s \leq s_1 + s_2$ which means $s = 0, 1, 2$.

The $s = 2, m_s = 2$ state is easy to write down in the notation specified.

$$|2, 2\rangle = |1, 1\rangle|1, 1\rangle$$

$$\sqrt{6 - 2}|2, 1\rangle = \sqrt{2 - 0}|1, 0\rangle|1, 1\rangle + \sqrt{2 - 0}|1, 1\rangle|1, 0\rangle$$

$$|2, 1\rangle = \sqrt{\frac{1}{2}}|1, 0\rangle|1, 1\rangle + \sqrt{\frac{1}{2}}|1, 1\rangle|1, 0\rangle$$

$$S_-|2, 1\rangle = (S_{1-} + S_{2-})\left(|1, 0\rangle|1, 1\rangle + |1, 1\rangle|1, 0\rangle\right)$$

$$\sqrt{6}|2, 0\rangle = \sqrt{\frac{1}{2}}\sqrt{2}|1, -1\rangle|1, 1\rangle + \sqrt{\frac{1}{2}}\sqrt{2}|1, 0\rangle|1, 0\rangle + \sqrt{\frac{1}{2}}\sqrt{2}|1, 0\rangle|1, 0\rangle + \sqrt{\frac{1}{2}}\sqrt{2}|1, 1\rangle|1, -1\rangle$$

$$|2, 0\rangle = \sqrt{\frac{1}{6}}|1, -1\rangle|1, 1\rangle + \sqrt{\frac{4}{6}}|1, 0\rangle|1, 0\rangle + \sqrt{\frac{1}{6}}|1, 1\rangle|1, 1\rangle$$

Unlike the previous problem, this problem asks us to find the states with lower values of $s$. We write down the state orthogonal to $|2, 1\rangle$ that has $m = 1$ then lower it to get $|2, 0\rangle$. Then we find the $|0, 0\rangle$ state by picking the $m = 0$ state orthogonal to $|2, 0\rangle$ and $|1, 0\rangle$.

$$|1, 1\rangle = \sqrt{\frac{1}{2}}|1, 0\rangle|1, 1\rangle - \sqrt{\frac{1}{2}}|1, 1\rangle|1, 0\rangle$$

$$|1, 0\rangle = \sqrt{\frac{1}{2}}|1, -1\rangle|1, 1\rangle - \sqrt{\frac{1}{2}}|1, 1\rangle|1, -1\rangle$$

$$|0, 0\rangle = \sqrt{\frac{1}{3}}|1, -1\rangle|1, 1\rangle - \sqrt{\frac{1}{3}}|1, 1\rangle|1, -1\rangle + \sqrt{\frac{1}{3}}|1, 1\rangle|1, 1\rangle$$
7. A hydrogen atom is in the state $\psi = R_{43}Y_{30}\chi_+$. A combined measurement of of $J^2$ and of $J_z$ is made. What are the possible outcomes of this combined measurement and what are the probabilities of each? You may ignore nuclear spin in this problem.

This is again an addition of angular momentum problem. We can read the quantum numbers from the hydrogen state $\psi = R_{nt}Y_{lmt}\chi_+ = \psi = R_{43}Y_{30}\chi_+$. We see that $\ell = 3$ and $m_\ell = 0$. Since this is hydrogen the electron has $s = \frac{1}{2}$ and the state $\chi_+$ means $m_s = \frac{1}{2}$. Since $m_j = m_\ell + m_s$, we know $m_j = \frac{1}{2}$. For $\vec{J} = \vec{L} + \vec{S}$, we have $|\ell - s| \leq j \leq \ell + s$ or $\frac{5}{2} \leq j \leq \frac{7}{2}$ or $j = \frac{5}{2},\frac{7}{2}$. So this gives us the two possible outcomes of the combined measurement.

$J^2 = j(j+1)\hbar^2 = \frac{63}{4}\hbar^2$ and $J_z = m_j\hbar = \frac{1}{2}\hbar$

or $J^2 = j(j+1)\hbar^2 = \frac{35}{4}\hbar^2$ and $J_z = m_j\hbar = \frac{1}{2}\hbar$

We have solved the problem of adding spin $\frac{1}{2}$ to some $\ell$ in general and have the state with $j = \ell + \frac{1}{2}$ given in the formulas.

$$\psi_{j, m_j} = \psi_{\ell + \frac{1}{2}, m + \frac{1}{2}} = \sqrt{\frac{\ell + m + 1}{2\ell + 1}} Y_{\ell m} \chi_+ + \sqrt{\frac{\ell - m}{2\ell + 1}} Y_{\ell m + 1} \chi_-$$

Comparing this to $\psi = R_{43}Y_{30}\chi_+$, we see that $m = 0$ in the formula. We get the probability to get $j = \frac{7}{2}$ by taking the dot product.

$$P_{\frac{7}{2}} = |\langle \sqrt{\frac{3 + 0 + 1}{7}} Y_{30} \chi_+ + \sqrt{\frac{3 - 0}{7}} Y_{31} \chi_- | Y_{30} \chi_+ \rangle|^2 = \frac{4}{7}$$

We can use the other similar formula to read of that the probability to measure $j = \frac{5}{2}$ is $P_{\frac{5}{2}} = \frac{3}{7}$.

8. Two (identical) electrons are bound in a Helium atom. What are the allowed states $|j\ell s\ell_1 \ell_2\rangle$ if both electrons have principal quantum number $n = 1$? What are the states if one has $n = 1$ and the other $n = 2$?

If an electron is in an $n = 1$ state, it can only have $\ell = 0$ since $\ell < n$ for hydrogen states. So the only angular momenta we can add is the two spins. Adding spin one-half to spin one-half gives us a spin 1 state that is symmetric under interchange and a spin-0 state that is antisymmetric under interchange. Remember that the highest angular momentum state is symmetric. But the overall wavefunction must be antisymmetric and the spatial states are both the same ($n = 1, \ell = 0$) so the spatial state is symmetric and we must choose the antisymmetric spin state. So we get only one possibility $j = \frac{1}{2}, \ell = 0, s = 0, \ell_1 = 0, \ell_2 = 0$.

If an electron is in the $n = 2$ state, we can have $\ell = 0, 1$. With the spatial states being different, we can have a symmetric or antisymmetric spatial state and therefore there is no constraint on the spin.

So for $\ell_1 = 0, \ell_2 = 0$, we must have $\ell = 0$. $s$ can be 0 or 1 and $j = s$.

For $\ell_1 = 0, \ell_2 = 1$, we have $\ell = 1$. If $s = 0$ then $j = 1$. If $s = 1$ then $j = 0, 1, 2$.

9. Assume an electron is bound to a heavy positive particle with a harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Calculate the energy shifts to all the energy eigenstates in an electric field $E$ (in the $x$ direction).
This is a time independent perturbation theory problem. The unperturbed problem is simply a one dimensional harmonic oscillator with \( E_n = \left( n + \frac{1}{2} \right) \hbar \omega \). The perturbing potential is \( H_1 = eE x \) for a constant electric field. The first order perturbation theory result gives zero since \( \langle n| x | n \rangle = 0 \). This can be seen either from the raising and lowering calculation since \( x \) will raise or lower the state and the dot product will then give zero or from the calculation using an integral since the integral over \( x \) will be odd. So we need to go to second order perturbation theory. There are no degenerate states.

The perturbation can be written in terms of the raising and lowering operators.

\[
H_1 = eE x = eE \sqrt{\frac{\hbar}{2m\omega}} (A + A^\dagger)
\]

The energy shift to the \( n \)th energy eigenstate to second order is.

\[
E_n^{(2)} = \sum_{k \neq n} \frac{|\langle k| H_1 | n \rangle|^2}{E_n^{(0)} - E_k^{(0)}} = \frac{e^2 E^2 \hbar}{2m\omega} \sum_{k \neq n} \frac{|\langle k| A + A^\dagger | n \rangle|^2}{(n - k)\hbar \omega} = \frac{e^2 E^2 \hbar}{2m\omega} \sum_{k \neq n} \frac{\sqrt{n} \delta_{k,n-1} + \sqrt{n} + 1 \delta_{k,n+1}}{(n - k)\hbar \omega}
\]

\[
E_n^{(2)} = \frac{e^2 E^2 \hbar}{2m\omega} \left( \frac{n}{\hbar \omega} - \frac{n + 1}{\hbar \omega} \right) = -\frac{e^2 E^2 \hbar}{2m\omega \hbar \omega} = -\frac{e^2 E^2}{2m^2 \omega^2}
\]

10. Calculate the fine structure energy shifts (in eV!) for the \( n = 1, n = 2, \) and \( n = 3 \) states of Hydrogen. Include the effects of relativistic corrections, the spin-orbit interaction, and the so-called Darwin term (due to Dirac equation). Do not include hyperfine splitting or the effects of an external magnetic field. (Note: I am not asking you to derive the equations.) Clearly list the states in spectroscopic notation and make a diagram showing the allowed electric dipole decays of these states.

This problem just asks you to show the results of the fine structure calculations that are given in the formulas.

\[
\Delta E_{12} = -\frac{1}{2n^3} \alpha^4 m c^2 \left( \frac{1}{j + \frac{1}{2}} - \frac{3}{4n} \right)
\]

The answer only depends on \( n \) and \( j \). The states to be calculated are the \( 1S_\frac{1}{2} \), \( 2S_\frac{1}{2} \) which will have the same energy as the \( 2P_\frac{1}{2} \), the \( 2P_\frac{3}{2} \), the \( 3S_\frac{1}{2} \) which will have the same energy as the \( 3P_\frac{1}{2} \), the \( 3P_\frac{3}{2} \) which will have the same energy as the \( 3D_\frac{3}{2} \), and the \( 3D_\frac{5}{2} \). Let me just do and the \( n = 2 \) example here.

\[
\Delta E_{12} = -\frac{\alpha^4 m c^2}{16} \left( \frac{1}{j + \frac{1}{2}} - \frac{3}{8} \right)
\]

First for \( j = \frac{1}{2} \).

\[
\Delta E_{12} = -\frac{\alpha^4 m c^2}{16} \left( \frac{5}{8} \right) = -\frac{5\alpha^4 m c^2}{128} = -0.0000566 eV
\]

Now for \( j = \frac{3}{2} \).

\[
\Delta E_{12} = -\frac{\alpha^4 m c^2}{16} \left( \frac{1}{2} - \frac{3}{8} \right) = -\frac{\alpha^4 m c^2}{128} = -0.0000113 eV
\]

See the class notes for the diagram.
11. Calculate and show the splitting of the $n = 3$ states (as in the previous problem) in a weak magnetic field $B$. Draw a diagram showing the states before and after the field is applied.

This problem asks you to use the calculation we did of splitting of Hydrogen in a weak $B$ field. The formula is given in the Hydrogen section.

$$\Delta E_B = \frac{e\hbar B}{2mc}(1 \pm \frac{1}{2\ell + 1})m_j$$

for $j = \ell \pm \frac{1}{2}$. This is to be added to the fine structure splitting from the problem above. Each $j$ state splits into $2j + 1$ states of different $m_j$. The energy shift from the formula is the familiar $g_L\mu_B B m_j$ where $g_L$ is the Lande g factor which multiplies the basic energy shift of $e\hbar B/2mc m_j = \mu_B B m_j$. The Lande $g$ factors are 2 for the $3S_\frac{1}{2}$, $\frac{2}{3}$ for the $3P_\frac{1}{2}$, $\frac{4}{3}$ for the $3P_\frac{3}{2}$, $\frac{4}{3}$ for the $3D_\frac{3}{2}$, and $\frac{6}{5}$ for the $3D_\frac{5}{2}$. See the class notes for a diagram.

12. The energies of photons emitted in the Hydrogen atom transition between the $3S$ and the $2P$ states are measured, first with no external field, then, with the atoms in a uniform magnetic field $B$. Explain in detail the spectrum of photons before and after the field is applied. Be sure to give an expression for any relevant energy differences.

This is a problem that again employs the two things we calculated in the fine structure chapter, the fine structure correction and the splitting in a weak magnetic field. The energy eigenstates are the total angular momentum states with $j = \ell \pm \frac{1}{2}$. Of course the S state can only have $j = \frac{1}{2}$. We therefore need these corrections and splittings for the $3S_\frac{1}{2}$, $2P_\frac{1}{2}$, and $2P_\frac{3}{2}$ states.

$$\Delta E_{12} = -\frac{1}{2n^3}\alpha^4 mc^2(\frac{1}{j + \frac{1}{2}} - \frac{3}{4n})$$

$$\Delta E_B = \frac{e\hbar B}{2mc}(1 \pm \frac{1}{2\ell + 1})m_j$$

for $j = \ell \pm \frac{1}{2}$.

For the $3S_\frac{1}{2}$ state,

$$E = -\frac{\alpha^2 mc^2}{18} - \frac{1}{54}\alpha^4 mc^2(\frac{3}{4}) + 2\frac{e\hbar B}{2mc} m_j$$

For the $2P_\frac{1}{2}$ state,

$$E = -\frac{\alpha^2 mc^2}{8} - \frac{1}{16}\alpha^4 mc^2(\frac{5}{8}) + 2\frac{e\hbar B}{32mc}(2)m_j$$

For the $2P_\frac{3}{2}$ state,

$$E = -\frac{\alpha^2 mc^2}{8} - \frac{1}{16}\alpha^4 mc^2(\frac{1}{8}) + 4\frac{e\hbar B}{32mc}(2)m_j$$

The energies of the photons emitted will be the energy differences between the states $E = E_{3S} - E_{2P}$. For the transitions to the $2P_\frac{1}{2}$ state,

$$E = \frac{5\alpha^2 mc^2}{72} + \left(\frac{5}{128} - \frac{1}{72}\right)\alpha^4 mc^2 + \frac{e\hbar B}{2mc}(2)(2m_{j1} - \frac{2}{3}m_{j2})$$

You should do the energy for transitions to the $2P_\frac{3}{2}$ too.
13. Calculate the energy shifts to the four hyperfine ground states of hydrogen in a weak magnetic field. (The field is weak enough so that the perturbation is smaller than the hyperfine splitting.)

This is the calculation we did in the section on hyperfine splitting. We did it for strong and intermediate \( B \) fields too but this is the weak \( B \) field case. The four states referred to in the problem are the hyperfine states with total spin \( f = 1 \) and \( f = 0 \). In the weak \( B \) field case we assume the \( f = 0 \) state is not degenerate with the others. We find the perturbation gives a diagonal matrix for the \( f = 1 \) states so the energy shifts are just the simple first order perturbation results. The calculation is done in the section “Hyperfine Splitting in a Weak \( B \) Field” in the notes.

14. Write down the ground state (in spectroscopic notation) for the element Oxygen \((Z = 8)\).

This exercises the use of Hund’s rules from atomic physics which I should give you in the formulas. The \( 1S \) and \( 2S \) take the first 4 electrons and there are 4 electrons in the \( 2P \) state. We can think of this as 2 holes in the \( 2P \) state. The first rule is maximum \( s \) which means \( s = 1 \) for the two holes. The next rule is maximum \( \ell \) which means \( \ell = 2 \) for the two holes. The shell is more than half full so we couple these to give maximum \( j \) so this is \( j = 3 \). The state would be labeled \( ^3D_3 \).
15. A hydrogen atom is in a uniform electric field in the z direction which turns on abruptly at \( t = 0 \) and decays exponentially as a function of time, \( E(t) = E_0 e^{-t/\tau} \). The atom is initially in its ground state. Find the probability for the atom to have made a transition to the \( 2P \) state as \( t \to \infty \). You need not evaluate the radial part of the integral. What \( z \) components of orbital angular momentum are allowed in the \( 2P \) states generated by this transition? This is just a time dependent perturbation problem. It is not a harmonic potential. We use the most basic time dependent perturbation formula.

\[
c_n(t) = \frac{1}{i\hbar} \int_0^t dt' e^{i(E_n-E) t'/\hbar} \langle \phi_n | V(t') | \phi_i \rangle
\]

The perturbing potential is \( V(t) = eE_0 e^{-t/\tau} z = eE_0 e^{-t/\tau} r \cos \theta = eE_0 e^{-t/\tau} r \sqrt{\frac{2}{3}} Y_{10} \). The initial state is the \( 1S \) state and the final state is the \( 2P \) state.

\[
c_{2p}(\infty) = \frac{eE_0}{i\hbar} \sqrt{\frac{4\pi}{3}} \int_0^\infty dt' e^{i(E_n-E) t'/\hbar} e^{-t'/\tau} \langle \psi_{21m} | rY_{10} | \psi_{100} \rangle
\]

\[
c_{2p} = \frac{eE_0}{i\hbar} \sqrt{\frac{1}{3}} \left[ e^{i(\omega_21-1/\tau)t'} \right]_{0}^{\infty} \int dr R_{21}^* r^3 R_{10} \delta_{m0}
\]

\[
c_{2p} = \frac{eE_0}{i\hbar} \sqrt{\frac{1}{3}} \left[ \frac{1}{i\omega_21-1/\tau} \right] \int dr \sqrt{\frac{1}{3}} \left( \frac{1}{2a_0} \right)^{3/2} e^{-r/2a_0} \frac{3}{3} e^{-r/2a_0}
\]

\[
c_{2p} = \frac{eE_0}{i\hbar} \frac{1}{3} \left( \frac{1}{2a_0} \right)^{3/2} e^{-r/2a_0}
\]

\[
c_{2p} = \frac{ieE_0}{3\sqrt{2} i \omega_21-1/\tau} \frac{1}{a_0} 6 \cdot 16
\]

\[
c_{2p} = \frac{512 e^2 E_0^2 a_0^2}{81^2 \left( \frac{2}{3} \alpha^2 mc^2 \right)^2 - \frac{\hbar^2}{\tau^2}}
\]

16. State the selection rules for radiative transitions between hydrogen atom states in the electric dipole approximation. These are rules for the allowed changes in \( l, m, s, \) and parity. They can be easily derived from the matrix element given on the front of the test. Draw an energy level diagram (up to \( n = 3 \)) for hydrogen atoms in a weak \( B \) field. Show the allowed E1 transitions from \( n = 3 \) to \( n = 1 \) on that diagram.
\[ h = 1.05 \times 10^{-27} \text{ erg sec} \quad c = 3.00 \times 10^{10} \text{ cm/sec} \quad e = 1.602 \times 10^{-19} \text{ coulomb} \]

\[ 1 \text{eV} = 1.602 \times 10^{-12} \text{ erg} \quad \alpha = \frac{e^2}{\hbar c} = 1/137 \quad h = 1973 \text{ eV } \text{Å}= 197.3 \text{ MeV} \text{F} \]

\[ 1 \text{Å} = 1.0 \times 10^{-8} \text{ cm} \quad 1 \text{Fermi} = 1.0 \times 10^{-13} \text{ cm} \quad a_0 = \frac{\hbar}{2m_e e} = 0.529 \times 10^{-8} \text{ cm} \]

\[ m_p = 938.3 \text{ MeV}/c^2 \quad m_n = 939.6 \text{ MeV}/c^2 \quad m_e = 9.11 \times 10^{-28} \text{ g} = 0.511 \text{ MeV}/c^2 \]

\[ k_B = 1.38 \times 10^{-16} \text{ erg}/^0\text{K} \quad g_e = 2 + \frac{\alpha}{\pi} \quad g_p = 5.6 \]

\[ \mu_{\text{Bohr}} = \frac{e\hbar}{2m_e c} = 0.579 \times 10^{-8} \text{ eV/ gauss} \]

\[ \int_{-\infty}^{\infty} dx \ f(x) \ \delta(x-a) = f(a) \quad \int_{-\infty}^{\infty} dx \ e^{-ax^2} = \sqrt{\frac{\pi}{a}} \]

\[ e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad \sin \theta = \sum_{n=1,3,5,...} \frac{g^n}{n!} (-1)^{n-1} \quad \cos \theta = \sum_{n=0,2,4,...} \frac{g^n}{n!} (-1)^2 \]

\[ P(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-x^2/2\sigma^2} \quad \int_0^{\infty} dr \ r^n e^{-ar} = \frac{n!}{a^{n+1}} \]

\[ E = \sqrt{m^2 c^4 + p^2 c^2} \]

**GENERAL WAVE MECHANICS**

\[ E = \hbar \nu = \hbar \omega \quad \lambda = \hbar / p \quad p = \hbar k \]

\[ \Delta p \Delta x \geq \frac{\hbar}{2} \quad \Delta A \Delta B \geq \frac{1}{2} \langle [A,B] \rangle \]

\[ \psi(x) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} dp \ \phi(p) e^{ipx/\hbar} \]

\[ \phi(p) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} dx \ \psi(x) e^{-ipx/\hbar} \]

\[ p_{op} = \hbar \frac{\partial}{\partial x} \quad E_{op} = i\hbar \frac{\partial}{\partial \nu} \quad x_{op} = i\hbar \frac{\partial}{\partial \phi} \]

\[ Hu_j(x) = E_j u_j(x) \quad \psi_j(x,t) = \psi_j(x) e^{-iE_j t/\hbar} \]

\[ \psi(x) \text{ continuous when } \frac{d\psi}{dx} \text{ continuous if } V \text{ finite} \]

\[ \Delta \frac{d\psi}{dx} = \frac{2m\lambda}{\hbar^2} \psi(a) \text{ for } V(x) = \lambda \delta(x-a) \]

\[ \langle \psi|\psi \rangle = \int_{-\infty}^{\infty} dx \phi^*(x) \psi(x) \quad \langle u_i|u_j \rangle = \delta_{ij} \quad \sum_i |u_i \rangle \langle u_i | = 1 \]

\[ \psi = \sum_i a_i u_i \quad a_i = \langle u_i|\phi \rangle \quad \psi(x) = \langle x|\psi \rangle \]

\[ \langle \psi|A|\psi \rangle = \langle \psi|A^* \phi \rangle^* = \langle \phi|A \psi \rangle = \langle A^\dagger \phi|\psi \rangle \quad \phi(p) = \langle p|\psi \rangle \]

\[ [\frac{1}{2m}(\vec{p} + \frac{e}{c} \vec{A})^2 + V(\vec{r})] \psi(\vec{r}) = E \psi(\vec{r}) \quad H \psi = E \psi \]

\[ [p_x, x] = \hbar \quad [L_x, L_y] = i\hbar L_z \quad [L^2, L_z] = 0 \]

\[ \psi_i = \langle u_i|\psi \rangle \quad A_{ij} = \langle u_i|A|u_j \rangle \]

\[ \frac{dA}{dt} = (\frac{\partial A}{\partial \nu}) + \frac{i}{\hbar} \langle [H, A] \rangle \]

**HARMONIC OSCILLATOR**

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = \hbar \omega A^\dagger A + \frac{1}{2} \hbar \omega \]

\[ E_n = (n + \frac{1}{2})\hbar \omega \quad n = 0, 1, 2... \]

\[ U_n(x) = \sum_{k=0}^{\infty} a_k y^k e^{-y^2/2} \quad a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k \quad y = \sqrt{\frac{m\omega}{\hbar}} x \]

\[ A = (\sqrt{\frac{m\omega}{2\hbar}} x + i \frac{p}{\sqrt{2m\hbar\omega}}) \quad A^\dagger = (\sqrt{\frac{m\omega}{2\hbar}} x - i \frac{p}{\sqrt{2m\hbar\omega}}) \quad [A, A^\dagger] = 1 \]

\[ A^\dagger |n \rangle = \sqrt{(n+1)} |n+1 \rangle \quad A |n \rangle = \sqrt{(n)} |n-1 \rangle \quad u_0(x) = (\frac{m\omega}{\hbar \pi})^{1/4} e^{-m\omega x^2/2\hbar} \]
**ANGULAR MOMENTUM**

<table>
<thead>
<tr>
<th>$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$</th>
<th>$[L^2, L_i] = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2Y_{\ell m} = \ell(\ell+1)\hbar^2Y_{\ell m}$</td>
<td>$L_xY_{\ell m} = m\hbar Y_{\ell m}$</td>
</tr>
<tr>
<td>$L_z = L_x \pm iL_y$</td>
<td>$L_{\pm}Y_{\ell m} = h\sqrt{\ell(\ell+1) - m(m \pm 1)} Y_{\ell,m\pm 1}$</td>
</tr>
<tr>
<td>$Y_{00} = \frac{1}{\sqrt{4\pi}}$</td>
<td>$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta$</td>
</tr>
<tr>
<td>$Y_{22} = -\sqrt{\frac{15}{92\pi}} \sin^2 \theta$</td>
<td>$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$</td>
</tr>
<tr>
<td>$Y_{21} = -\sqrt{\frac{15}{92\pi}} \sin \theta \cos \theta$</td>
<td>$Y_{20} = \sqrt{\frac{3}{10\pi}} (3\cos^2 \theta - 1)$</td>
</tr>
<tr>
<td>$Y_{0(-m)} \equiv (-1)^\ell Y^*_{\ell m}$</td>
<td></td>
</tr>
</tbody>
</table>

\[ \psi_{n\ell m}(r) = Y_{\ell m}(\theta, \phi) e^{-\frac{1}{2}mc^2r^2} \]

**HYDROGEN ATOM**

$H = \frac{\hbar^2}{2\mu} - \frac{Ze^2}{r}$

$\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta, \phi)$

$E_n = -\frac{Z^2\alpha^2mc^2}{2n^2} = -\frac{13.6}{n^2} \text{eV}$

$n = n_r + \ell + 1$

$\ell = 0, 1, 2, 3, \ldots$

\[ R_{n\ell}(\rho) = \rho^\ell \sum_{k=0}^{\infty} a_k \rho^k e^{-\rho^2/2} \]

$a_{k+1} = \frac{k+\ell+1-n}{(k+1)(k+2)} a_k$

$\rho = \sqrt{-8\mu E} r$

$R_{10} = 2(\frac{Z}{2a_0})^2 \frac{\hbar}{a_0}$

$R_{20} = 2(\frac{Z}{2a_0})^2 (1 - \frac{Zr}{2a_0}) e^{-\frac{Zr}{2a_0}}$

$R_{21} = \frac{1}{\sqrt{3}} (\frac{Z}{2a_0})^2 (\frac{Zr}{a_0}) e^{-\frac{Zr}{2a_0}}$

$H_1 = -\frac{\hbar^2}{8\mu mc^2} \hat{S} \cdot \hat{L}$

$H_2 = \frac{\hbar^2}{2mrc^2} \hat{S} \cdot \hat{L}$

$\Delta E_{12} = -\frac{1}{2}\alpha^2 mc^2 (1 + \frac{3}{\ell + \frac{1}{2}} - \frac{3}{4\ell})$

$H_B = \frac{\hbar B}{2mc} (L_z + 2S_z)$

$\Delta E_B = \frac{\hbar B}{2mc} (1 \pm \frac{1}{2\ell+1}) |j_m| \text{ for } j = \ell \pm \frac{1}{2}$

**ADDITION OF ANGULAR MOMENTUM**

$\hat{J} = \hat{L} + \hat{S}$

$|l - s| \leq j \leq l + s$

$L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$

$\psi_{j m_l s} = \sum_{m_{\ell m s}} C(j m_l; lm_l sm_l) \psi_{lm_l \chi_{sm_l}} = \sum_{m_{\ell m s}} \langle j m_l s | lm_l sm_l \rangle \psi_{lm_l \chi_{sm_l}}$

$\psi_{j m_l} = \psi_{l+\frac{1}{2} m_l + \frac{1}{2}} \sqrt{\frac{\ell - m + 1}{2\ell + 1}} Y_{\ell m} \chi_+ + \sqrt{\frac{\ell - m}{2\ell + 1}} Y_{\ell m+1} \chi_- \quad \text{for } s = \frac{1}{2} \text{ and any } \ell$

$\psi_{j m_l} = \psi_{l-\frac{1}{2} m_l - \frac{1}{2}} \sqrt{\frac{\ell - m}{2\ell + 1}} Y_{\ell m} \chi_+ - \sqrt{\frac{\ell + m + 1}{2\ell + 1}} Y_{\ell m+1} \chi_- \quad \text{for } s = \frac{1}{2} \text{ and any } \ell$
PERTURBATION THEORY AND RADIATIVE DECAYS

\[ E_n^{(1)} = \langle \phi_n | H_1 | \phi_n \rangle \]
\[ E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \phi_n | H_1 | \phi_n \rangle|^2}{E_n^{(0)} - E_k^{(0)}} \]
\[ c_{nk} = \frac{\langle \phi_k | H_1 | \phi_n \rangle}{E_n^{(0)} - E_k^{(0)}} \]

\[ c_n(t) = \frac{i}{\hbar} \int_0^t dt' e^{i(E_n - E_k)t'/\hbar} \langle \phi_n | V(t') | \phi_k \rangle \]

Fermi’s Golden Rule:
\[ \Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle \psi_f | V | \psi_i \rangle|^2 \rho_f(E) \]
\[ \Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} \int \prod_k |Vd^3p_k| |M_{fi}|^2 \delta^3(\text{momentum conservation}) \delta(\text{Energy conservation}) \]

\[ \Gamma^{\text{rad}}_{m \rightarrow k} = \frac{\alpha^2}{2\pi mc^2} \int d\Omega_{\mu\omega_{km}} |\langle \phi_m | e^{-i\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} | \phi_k \rangle}|^2 \]
\[ \Gamma^{\text{rot}}_{m \rightarrow k} = \frac{\alpha^2}{2\pi c^2} \int d\Omega_{\mu\omega_{3km}} |\langle \phi_m | \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} | \phi_k \rangle|^2 \]
\[ \Delta l = \pm 1, \Delta s = 0 \]
\[ \hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = 0 \]
\[ I(\omega) \propto \frac{\Gamma^{1/2}}{\omega - \omega_0} \Gamma_{\text{collision}} = P \sigma \sqrt{\frac{3}{mkT}} \left( \frac{\Delta \omega}{\omega} \right)_{\text{Doppler}} = \sqrt{\frac{kT}{mc^2}} \]

ATOMS AND MOLECULES

Hund:
1) max \( s \)
2) max \( \ell \) (allowed)
3) min \( j \) (\( \leq \frac{1}{2} \) shell) else max \( j \)

\[ E_{\text{rot}} = \frac{\ell(\ell+1)\hbar^2}{2I} \approx \frac{1}{2000} \text{ eV} \]
\[ E_{\text{vib}} = (n + \frac{1}{2}) \hbar \omega \approx \frac{1}{50} \text{ eV} \]