NAME | Score

\[ \hbar = 1.05 \times 10^{-27} \text{ erg sec} \quad c = 3.00 \times 10^{10} \text{ cm/sec} \quad e = 1.602 \times 10^{-19} \text{ coulomb} \]

\[ 1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg} \quad \alpha = \frac{e^2}{4\pi \epsilon_0} = 1/137 \quad \hbar c = 1973 \text{ eV Å} = 197.3 \text{ MeV F} \]

\[ 1 \text{ Å} = 1.0 \times 10^{-8} \text{ cm} \quad 1 \text{ Fermi} = 1.0 \times 10^{-13} \text{ cm} \quad a_0 = \frac{\hbar}{\alpha m_e} = 0.529 \times 10^{-8} \text{ cm} \]

\[ m_p = 938.3 \text{ MeV}/c^2 \quad m_n = 939.6 \text{ MeV}/c^2 \quad m_e = 9.11 \times 10^{-28} \text{ g} = 0.511 \text{ MeV}/c^2 \]

\[ k_B = 1.38 \times 10^{-26} \text{ erg}/^\circ \text{K} \quad g_e = 2 + \frac{a}{x} \quad g_p = 5.6 \]

\[ \mu_{\text{Bohr}} = \frac{e\hbar}{2m_e c} = 0.579 \times 10^{-8} \text{ eV/gauss} \]

\[ \int_{-\infty}^{\infty} dx \ f(x) \delta(x - a) = f(a) \quad \int_{-\infty}^{\infty} dx \ e^{-ax^2} = \sqrt{\frac{\pi}{a}} \quad \text{use } \frac{\partial}{\partial \theta} \text{ for other forms} \]

\[ e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad \sin \theta = \sum_{n=1,3,5,...}^{\infty} \frac{\theta^n}{n!} (-1)^{n-1} \quad \cos \theta = \sum_{n=0,2,4,...}^{\infty} \frac{\theta^n}{n!} (-1)^{n/2} \]

\[ P(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-x^2/(2\sigma^2)} \quad \int_0^\infty dr \ r^ne^{-ar} = \frac{n!}{a^{n+1}} \quad E = \sqrt{m^2 c^4 + p^2 c^2} \]

**GENERAL WAVE MECHANICS**

\[ E = h\nu = \hbar \omega \]

\[ \lambda = h/p \quad p = \hbar k \]

\[ \Delta p \Delta x \geq \frac{\hbar}{2} \quad \Delta A \Delta B \geq \left\{ \frac{1}{\pi} [A, B] \right\} \quad \Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \]

\[ \psi(x) = \frac{1}{\sqrt{2\pi}\hbar} \int dp \phi(p) e^{ipx/\hbar} \]

\[ \phi(p) = \frac{1}{\sqrt{2\pi}\hbar} \int dx \psi(x) e^{-ipx/\hbar} \]

\[ p_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad E_{op} = i\hbar \frac{\partial}{\partial t} \quad x_{op} = i\hbar \frac{\partial}{\partial p} \]

\[ \hat{H} u_j(x) = E_j u_j(x) \quad \psi_j(x,t) = u_j(x)e^{-iE_j t/\hbar} \quad -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = i\hbar \frac{\partial \psi}{\partial t} \]

\[ \psi(x) \text{ continuous} \quad \frac{d\psi}{dx} \text{ continuous if } V \text{ finite} \]

\[ \Delta \frac{\partial \psi}{\partial x} = \frac{2m\lambda}{\hbar^2} \psi(a) \text{ for } V(x) = \lambda \delta(x - a) \]

\[ \langle \phi|\psi \rangle = \int_{-\infty}^{\infty} dx \phi^*(x)\psi(x) \quad \langle u_i|u_j \rangle = \delta_{ij} \quad \sum_i |u_i\rangle \langle u_i| = 1 \]

\[ \phi = \sum_i a_i u_i \quad a_i = \langle u_i|\phi \rangle \quad \psi(x) = \langle x|\psi \rangle \]

\[ \langle \phi|A|\psi \rangle = \langle \psi|A^\dagger|\phi \rangle = \langle \psi|A\phi \rangle = \langle \phi|A^\dagger|\psi \rangle \quad \phi(p) = \langle p|\psi \rangle \]

\[ H\psi = E\psi \]

\[ [p_x, x] = \frac{\hbar}{i} \quad [L_x, L_y] = i\hbar L_z \quad [L^2, L_z] = 0 \]

\[ \psi_i = \langle u_i|\psi \rangle \quad A_{ij} = \langle u_i|A|u_j \rangle \quad \frac{d(A)}{dt} = \frac{i}{\hbar} \langle [H, A] \rangle \]
HARMONIC OSCILLATOR

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar \omega A^\dagger A + \frac{1}{2}\hbar \omega \]

\[ u_n(x) = \sum_{k=0}^{\infty} a_k y^k e^{-y^2/2} \quad a_{k+2} = \frac{2(k-n)}{(k+1)(k+2)} a_k \]

\[ A = \left( \sqrt{\frac{m\omega}{2\hbar}} x + i \frac{p}{\sqrt{2m\hbar}} \right) \quad A^\dagger = \left( \sqrt{\frac{m\omega}{2\hbar}} x - i \frac{p}{\sqrt{2m\hbar}} \right) \quad [A, A^\dagger] = 1 \]

\[ A^\dagger |n\rangle = (n+1) |n+1\rangle \quad A |n\rangle = \sqrt{(n-1)} |n-1\rangle \]

\[ u_0(x) = \left( \frac{m\omega}{\hbar} \right)^{1/4} e^{-m\omega x^2/2\hbar} \]

ANGULAR MOMENTUM

\[ [L_i, L_j] = i\hbar \epsilon_{ijk} L_k \quad [\hat{L}^2, L_i] = 0 \]

\[ L^2 Y_{\ell m} = \ell (\ell + 1) \hbar^2 Y_{\ell m} \quad L_z Y_{\ell m} = \hbar Y_{\ell m} \]

\[ \hat{L}_\pm = L_x \pm i L_y \quad L_\pm Y_{\ell m} = \hbar \sqrt{\ell (\ell + 1) - m (m \pm 1)} Y_{\ell,m \pm 1} \]

\[ Y_{\ell 0} = \frac{1}{\sqrt{2\pi}} \sin \theta \quad Y_{11} = -\frac{3}{4\pi} e^{i\phi} \sin \theta \quad Y_{10} = \frac{3}{4\pi} \cos \theta \]

\[ Y_{22} = \frac{1}{9} \sin^2 \theta \quad Y_{21} = -\frac{15}{8\pi} e^{i\phi} \sin \theta \cos \theta \quad Y_{20} = \frac{5}{36\pi} (3 \cos^2 \theta - 1) \]

\[ Y_{20} = (-1)^\ell Y_{\ell m} \quad Y_{\ell m} (\pi - \theta, \phi + \pi) = (-1)^\ell Y_{\ell m} (\theta, \phi) \]

\[ \frac{-\hbar^2}{2\mu} \left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right] R_{n\ell}(r) + \left( V(r) + \frac{\ell (\ell + 1) \hbar^2}{2\mu r^2} \right) R_{n\ell}(r) = E R_{n\ell}(r) \]

\[ j_0 (kr) = \frac{\sin(kr)}{kr} \quad n_0(kr) = -\frac{\cos(kr)}{kr} \quad h_{\ell}^{(1)}(kr) = j_\ell(kr) + in_\ell(kr) \]

\[ H = H_0 - \vec{\mu} \cdot \vec{B} \quad \vec{\mu} = \frac{e}{2mc} \vec{L} \quad \vec{\mu} = \frac{e}{2mc} \vec{S} \]

\[ S_i = \frac{n}{2} \sigma_i \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

HYDROGEN ATOM

\[ H = \frac{p^2}{2\mu} - \frac{Ze^2}{r} \quad \psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \quad E_n = -\frac{Z^2 e^2 \mu c^2}{2n^2} = -\frac{13.6}{n^2} \text{ eV} \]

\[ n = n_r + \ell + 1 \]

\[ a_0 = \frac{n}{\alpha \mu c} \quad \ell = 0, 1, \ldots , n - 1 \]

\[ R_{n\ell}(\rho) = \rho^{\ell+1} \sum_{k=0}^{n-\ell} \frac{a_k \rho^k e^{-\rho^2/2}}{(k+1)!} \quad a_{k+1} = \frac{k+\ell+1-n}{(k+1)(k+2+\rho^2)} a_k \]

\[ \rho = \sqrt{\frac{8n E_r}{\hbar^2}} = \frac{2r}{\alpha_n} \]

\[ R_{20} = 2 \left( \frac{Z}{2\alpha_0} \right)^2 \left( 1 - \frac{Zr}{2\alpha_0} \right) e^{-\frac{Zr}{2\alpha_0}} \quad R_{21} = \frac{1}{\sqrt{2}} \left( \frac{Z}{2\alpha_0} \right)^2 \left( \frac{Zr}{\alpha_0} \right) e^{-\frac{Zr}{2\alpha_0}} \]

\[ R_{n,n-1} \propto r^{n-1} e^{-Zr/\alpha_0} \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \quad \langle \psi_{n\ell m} | \frac{e^2}{r} | \psi_{n\ell m} \rangle = \frac{Z^2 e^2}{n^2 \alpha_0^2} = \frac{2Z^2 e^2 \mu c^2}{n^2} \]
SHORT PROBLEMS (5 points each)

1. Use the uncertainty principle to estimate the binding energy of a neutron to a nucleus. Assume the nucleus provides an approximately constant potential of -20 MeV for $r < 2$ fermis and zero potential for $r > 2$.

2. A laser beam containing photons of energy 2 eV is incident upon two narrow slits separated by a distance of 0.1 mm. Find the angles (in radians) of the first three minima in the diffraction pattern.

3. At $t = 0$, a particle of mass $m$ is in the second excited state ($u_2$) of a 1D Harmonic Oscillator $(H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2)$. Find the expected value of $p^2$ as a function of time.

4. Find the normalized second excited state of the harmonic oscillator $u_2(x)$ in position space.

5. Find the eigen-energies for bound states in the potential $V(x) = -D\delta(x)$, where $D$ is a positive constant.

6. Light from a distant galaxy passes through a cloud of cool Hydrogen gas (in the ground state). Give a formula for the energies of light that will be absorbed by the gas.

7. A particle of mass $m$ is in the 3D potential $V(r) = \frac{1}{2}m\omega^2r^2$. The solution of the differential equation shows that the radial wave functions and energies are given by the equations below with $y = \sqrt{\frac{m\omega}{\hbar}}r$.

$$R_{n,\ell} = y^\ell \sum_{k=0}^{\infty} a_k y^{2k} e^{-y^2/2}$$

$$a_{k+1} = \frac{(k - n_r)}{(k + 1)(k + \ell + 3/2)} a_k$$

$$E = \left( 2n_r + \ell + \frac{3}{2} \right) \hbar \omega$$

Find the four lowest energies of this system. List the 3 quantum numbers describing all the states at each of these energies. How many degenerate states are there at each energy?

8. The hydrogen atom is made up of a proton and an electron bound together by the Coulomb force. The electron has a mass of 0.51 MeV/$c^2$. It is possible to make a hydrogen-like atom from a proton and a muon. The force binding the muon to the proton is identical to that for the electron but the muon has a mass of 106 MeV/$c^2$.

a) What is the ground state energy of muonic hydrogen (in eV).

b) What is the “Bohr Radius” of the ground state of muonic hydrogen (in Angstroms).

9. An electron in the Coulomb field of a proton is in the state described by the wave function $\frac{1}{2}(\psi_{100} + \psi_{211} - \psi_{210} + i\psi_{211-1})$. Find the expected value of the Energy, $L^2$ and $L_z$. Now find the expected value of $L_x$. 


10. The component of an electron’s spin along the $\hat{v}$ direction is measured. The operator for the $\hat{v}$ component of spin is $S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$. What are the possible outcomes of the measurement? What are the eigenstates corresponding to those outcomes?

Long Problems (10 points)

1. Prove that, if the Hamiltonian is symmetric under rotations, then $[H, L_z] = 0$. Use this to show that $[H, L^2] = 0$.

2. At $t = 0$, an electron in the Coulomb field of a proton is in the state described by the wave function $\frac{1}{\sqrt{2}}(\psi_{100} + i \psi_{200})$. Find the expected value of $r$ as a function of time.

3. A electron is in an eigenstate of $S_y$ with eigenvalue $\frac{\hbar}{2}$ at $t = 0$. The particle is in a magnetic field $\vec{B} = B \hat{z}$ which makes the Hamiltonian for the system $H = \mu_B B \sigma_z$. Find the probability to measure $S_y = -\frac{\hbar}{2}$ as a function of time.