Midterm **Nov. 8, 2006** **Physics 130B**

1. At $t = 0$, an electron is the spin state

$$\psi(t=0) = \begin{pmatrix} i\sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}.$$

A magnetic field B is applied in the z direction. **a)** Find the spin state of the particle as a function of time. **b)** Find the expectation value of S_y as a function of time. **c)** What is the probability to measure that the electron's spin along the x direction is $\frac{\hbar}{2}$ as a function of time? *(10 points)*

$$H = -\vec{\mu} \cdot \vec{B} = \mu_B B \sigma_z$$

a) The eigenstates are then spin up and spin down along the z direction since these are the eigenstates of σ_z. The energies for spin up and down along the z direction are $\pm\mu_B B$. So it is easy to write $\psi(t)$ since the two components of the spinor represent the energy eigenstates. Let $\omega = \frac{\mu_B B}{\hbar}$.

$$\psi(t) = \begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix}.$$

b) Simply compute the expectation value in this state.

$$\langle S_y \rangle = \frac{\hbar}{2} \begin{pmatrix} -i\sqrt{\frac{2}{3}}e^{i\omega t} & \sqrt{\frac{1}{3}}e^{-i\omega t} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} -i\sqrt{\frac{2}{3}}e^{i\omega t} & \sqrt{\frac{1}{3}}e^{-i\omega t} \end{pmatrix} \begin{pmatrix} -i\sqrt{\frac{1}{3}}e^{i\omega t} \\ -\sqrt{\frac{2}{3}}e^{-i\omega t} \end{pmatrix}$$

$$\langle S_y \rangle = \frac{\hbar}{2}\left[-\frac{\sqrt{2}}{3}e^{i2\omega t} - \frac{\sqrt{2}}{3}e^{-i2\omega t}\right] = -\frac{\sqrt{2}}{3}\frac{\hbar}{2}2\cos(2\omega t) = -\frac{\sqrt{2}}{3}\hbar\cos(2\omega t)$$

c) To get the amplitude to have spin up in the x direction, dot with that eigenstate $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$$P = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix} \right|^2 = \left| i\sqrt{\frac{1}{3}}e^{-i\omega t} + \sqrt{\frac{1}{6}}e^{i\omega t} \right|^2 = \frac{1}{3} + \frac{1}{6} + i\sqrt{\frac{1}{3}}\sqrt{\frac{1}{6}}e^{-2i\omega t} - i\sqrt{\frac{1}{3}}\sqrt{\frac{1}{6}}e^{2i\omega t}$$

$$= \frac{1}{2} + \sqrt{\frac{1}{18}}2\sin(2\omega t) = \frac{1}{2} + \sqrt{\frac{2}{9}}\sin(2\omega t)$$

2. Consider a two dimensional harmonic oscillator problem described by the Hamiltonian

$$H_0 = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2).$$

Remember that this unperturbed problem can be easily solved in Cartesian coordinates. **a)** Accurately calculate the energy shifts to the degenerate first excited states, to first order, if the additional potential $V = \beta xy$ is applied. **b)** What are the new energy eigenstates for the degenerate first excited state (in terms of the standard eigenstates $u_0^{(x)} u_1^{(y)}$ and $u_1^{(x)} u_0^{(y)}$)? *(10 points)*

As stated in the problem, the degenerate first excited states are $u_1^{(x)} u_0^{(y)} = |10\rangle$ and $u_0^{(x)} u_1^{(y)} = |01\rangle$. The excitation in x has the same energy as the excitation in y. The perturbation V can be written in terms of the raising and lowering operators for x and y (which are not the same thing).

$$V = \beta xy = \frac{\beta\hbar}{2m\omega} \left(A + A^\dagger\right)_x \left(A + A^\dagger\right)_y$$

We need to compute the 2 by 2 matrix for V for these two states. The only non-zero matrix elements are

$$\langle 01|V|10\rangle = \langle 10|V|01\rangle = \frac{\beta\hbar}{2m\omega} \langle 01|A_{(x)} A_{(y)}^\dagger|10\rangle = \frac{\beta\hbar}{2m\omega}$$

$E_\pm = 2\hbar\omega \pm \frac{\beta\hbar}{2m\omega}$ for the energy eigenstates $\psi_\pm = \frac{1}{\sqrt{2}} (u_0 u_1 \pm u_1 u_0)$.

3. Two non-identical spin $\frac{1}{2}$ particles have a Hamiltonian given by

$$H = E_0 + B\vec{S}_1 \cdot \vec{S}_2 + C(S_{1z} + S_{2z}).$$

There are no spatial coordinates for this problem. Find the allowed energies and the energy eigenstates in terms of the four product states $\chi_+^{(1)}\chi_+^{(2)}$, $\chi_+^{(1)}\chi_-^{(2)}$, $\chi_-^{(1)}\chi_+^{(2)}$, and $\chi_-^{(1)}\chi_-^{(2)}$. *(10 points)*

The eigenstates of the the total spin operator S^2 will be the eigenstates of this Hamiltonian since $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$. The energies are $E = E_0 + B\frac{\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right) + C\hbar m_s$ for the states $|sm_s\rangle$.

$$|11\rangle = \chi_+\chi_+ \quad E_0 + \frac{B\hbar^2}{4} + C\hbar$$

$$|1-1\rangle = \chi_-\chi_- \quad E_0 + \frac{B\hbar^2}{4} - C\hbar$$

$$|10\rangle = \frac{1}{\sqrt{2}}(\chi_+\chi_- + \chi_-\chi_+) \qquad E_0 + \frac{B\hbar^2}{4}$$

$$|00\rangle = \frac{1}{\sqrt{2}}(\chi_+\chi_- - \chi_-\chi_+) \qquad E_0 - \frac{3B\hbar^2}{4}$$