Midterm **Oct. 22, 2003** **Physics 130B**

Name__________________________ **Secret Number**__________

1. A hydrogen atom is in the state $\psi = R_{21}Y_{10}\chi_-$. Let $\vec{J} = \vec{L} + \vec{S}$ be the total angular momentum operator; the sum of orbital and spin angular momenta. If a measurement of J_z is made, what are the possible outcomes of this measurement and what are the probabilities for each outcome? If a measurement of J^2 is made, what are the possible outcomes of this measurement and what are the probabilities for each outcome? *(10 points)*

   Solution:

   $\hat{J}_z = \hat{L}_z + \hat{S}_z$ here $\hat{J}_z\psi = -\frac{\hbar}{2}\psi$, so the outcome of measurement of J_z is $-\frac{\hbar}{2}$, the probability is 1.

   $l = 1$, $s = \frac{1}{2}$, so $j = \frac{1}{2}, \frac{3}{2}$, so the outcomes of measurement J^2 are: $\frac{1}{2}\frac{3}{2}\hbar^2 = \frac{3}{4}\hbar^2$, and $\frac{3}{2}\frac{5}{2}\hbar^2 = \frac{15}{4}\hbar^2$

   $\phi_{\frac{3}{2},-\frac{1}{2}} = \phi_{1+\frac{1}{2},-1+\frac{1}{2}} = \sqrt{\frac{1}{3}}Y_{1,-1}\chi_+ + \sqrt{\frac{2}{3}}Y_{1,0}\chi_-$

   $\phi_{\frac{1}{2},-\frac{1}{2}} = \phi_{1-\frac{1}{2},-1+\frac{1}{2}} = \sqrt{\frac{2}{3}}Y_{1,-1}\chi_+ - \sqrt{\frac{1}{3}}Y_{1,0}\chi_-$

   the probability of $\frac{3}{4}\hbar^2$ is $P = |<\phi_{\frac{1}{2},-\frac{1}{2}}|\psi>|^2 = \frac{1}{3}$

   the probability of $\frac{15}{4}\hbar^2$ is $P = |<\phi_{\frac{3}{2},-\frac{1}{2}}|\psi>|^2 = \frac{2}{3}$

2. A harmonic oscillator has a small anharmonic term such that the full Hamiltonian is

   $$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x^4.$$

   Use perturbation theory to estimate (all) the eigenenergies of the system. *(10 points)*

   Solution:

   $H = H_0 + H_1$, $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$, $H_1 = \lambda x^4$.

   so $E_n^0 = (n + \frac{1}{2})\hbar\omega$

   from perturbation theory $E_n^1 =< \phi_n^0|H_1|\phi_n^0 >=< n|\lambda x^4|n >$

   $x = (A + A^\dagger)\sqrt{\frac{\hbar}{2m\omega}}$

   $x^4 = (A + A^\dagger)^4\frac{\hbar^2}{4m^2\omega^2}$

   $(A + A^\dagger)^4 = (A^2 + A^{\dagger^2} + AA^\dagger + A^\dagger A)^2 = (A^2 + A^{\dagger^2} + 1 + 2A^\dagger A)^2$

   $E_n^1 =< n|\lambda x^4|n >= \frac{\hbar^2\lambda}{4m^2\omega^2} < n|(A^2 + A^{\dagger^2} + 1 + 2A^\dagger A)^2|n >= \frac{\hbar^2\lambda}{4m^2\omega^2} < n|(A^2A^{\dagger^2} + A^{\dagger^2}A^2 + 4A^\dagger AA^\dagger A + 4A^\dagger A + 1)|n >$

   $A^2A^{\dagger^2} = A(A^\dagger A+1)A^\dagger = AA^\dagger AA^\dagger + AA^\dagger = (A^\dagger A+1)(A^\dagger+1) + A^\dagger A+1 = A^\dagger AA^\dagger A + 2A^\dagger A + 1 + A^\dagger A + 1$

   so $E_n^1 = \frac{\hbar^2\lambda}{4m^2\omega^2}(6n^2 + 6n + 3)$

   $E_n = E_n^0 + E_n^1 = (n + \frac{1}{2})\hbar\omega + \frac{\hbar^2\lambda}{4m^2\omega^2}(6n^2 + 6n + 3)$

3. Consider the problem of a charged particle in an external magnetic in the z direction with the gauge chosen so that $\vec{A} = (-yB, 0, 0)$.

   - What are the symmetries of this problem and hence what are the constants of the motion?
   - Write the Schrödinger equation for this problem.
   - Use the constants of the motion to simplify the Schrödinger equation to a differential equation in one variable.
   - Write this equation clearly in the form of the Schrödinger equation for a harmonic oscillator.
   - Now find the eigenenergies of this equation in terms of one quantum number and the constants of the motion.

   (10 points)

   Solution:

   $\vec{A} = (-yB, 0, 0)$

   $H = \frac{1}{2m}(\vec{P} - \frac{q}{c}\vec{A})^2 = \frac{1}{2m}((p_x + \frac{q}{c}yB)^2 + p_y^2 + p_z^2) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2 + \frac{q^2}{c^2}y^2B^2 + \frac{2q}{c}yp_xB)$

   so $[H, p_x] = 0$, $[H, p_z] = 0$

   x,z have translational symmetry, so the constants of motion are p_x, p_z

   Schroedinger equation: $E\phi = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2 + \frac{q^2}{c^2}y^2B^2 + \frac{2q}{c}yp_xB)\phi$

   $\phi = u(y)e^{ik_xx}e^{ik_zz}$

   so we get: $-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}u(y) + (\frac{1}{2}m(\frac{qB}{mc})^2)^2(-y - \frac{\hbar ck_x}{qB})^2u(y) = (E - \frac{\hbar^2k_z^2}{2m})u(y)$

   this is the same as 1D harmonic oscillator with $\omega = \frac{qB}{mc}$ and $y_0 = -\frac{\hbar ck_x}{qB}$

   $E = (n + \frac{1}{2})\hbar\omega = \frac{\hbar qB}{mc}(n + \frac{1}{2}) + \frac{\hbar^2k_z^2}{2m}$