2. Ground State : $\psi_1 = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right)$; 1st excited state $\psi_2 = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right)$; 2nd excited state $\psi_3 = \frac{1}{\sqrt{2}} \cos\left(\frac{3\pi x}{2a}\right)$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n = E_n \psi_n \Rightarrow E_n^{(0)} = \frac{n^2\hbar^2\pi^2}{8ma^2} \qquad E_n = E_n^{(0)} + E_n^{(1)}$$

$$H_1 = \lambda \cos\left(\frac{\pi x}{2b}\right)$$

$$E_2^{(1)} = \langle \psi_2 | H_1 | \psi_2 \rangle = \frac{\lambda}{a}\int_{-a}^{a} \cos\left(\frac{\pi x}{2b}\right) \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{32 b^3 \lambda \sin\left(\frac{a\pi}{2b}\right)}{a(16b^2 - a^2)\pi}$$

$$E_3^{(1)} = \langle \psi_3 | H_1 | \psi_3 \rangle = \frac{\lambda}{a}\int_{-a}^{a} \cos\left(\frac{\pi x}{2b}\right) \cos^2\left(\frac{3\pi x}{2a}\right) dx = \frac{72\lambda b^3 \sin\left(\frac{a\pi}{2b}\right)}{a(36b^2 - a^2)\pi}$$

5. $[P, x] = \frac{\hbar}{i} \qquad H = \frac{P^2}{2m} + V \qquad [H, x] = -i\frac{\hbar}{m}P$

$$\sum_n (E_n - E_a)\, |\langle n|x|a\rangle|^2 = \frac{1}{2}\sum_n (2E_n - 2E_a)\langle a|x|n\rangle\langle n|x|a\rangle$$

$$= \frac{1}{2}\sum_n \left(\langle a|xH|n\rangle\langle n|x|a\rangle - \langle a|Hx|n\rangle\langle n|x|a\rangle + \langle a|x|n\rangle\langle n|Hx|a\rangle - \langle a|x|n\rangle\langle n|xH|a\rangle \right)$$

$$= \frac{1}{2}\sum_n \left(-\langle a|[H,x]|n\rangle\langle n|x|a\rangle + \langle a|x|n\rangle\langle n|[H,x]|a\rangle \right)$$

$$= \frac{1}{2}\langle a|[[H,x],x]|a\rangle$$

$$= \frac{-1}{2}\langle a|[\tfrac{i\hbar}{m}P, x]|a\rangle$$

$$= \frac{i\hbar}{2m}\langle a|\frac{\hbar}{i}|a\rangle = \frac{\hbar^2}{2m}$$

6. $H_0 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 r^2 \Rightarrow V(r) = \frac{1}{2}m\omega^2 r^2 \Rightarrow \frac{1}{r}\frac{dV}{dr} = m\omega^2$

$$H_{so} = \frac{1}{2m^2c^2}\vec{L}\cdot\vec{S}\,\frac{1}{r}\frac{dV}{dr} = \frac{\omega^2}{2mc^2}\,\frac{J^2 + L^2 - S^2}{2} = \frac{\omega^2\hbar^2}{4mc^2}[j(j+1) - l(l+1) - s(s+1)]$$

$$\Rightarrow E^{(1)} = \langle \psi | H_{so} | \psi \rangle = \frac{\omega^2\hbar^2}{4mc^2}[j(j+1) - l(l+1) - s(s+1)]$$

For relativistic correction $H_1 = -\frac{P^4}{8m^3c^2} = \frac{-1}{8m^3c^2}(P^2)^2$

$P^2 = P_x^2 + P_y^2 + P_z^2 \qquad P_x = i\sqrt{\frac{m\hbar\omega}{2}}(A_x^+ - A_x)$ similar for P_y, P_z $\qquad$ let $|000\rangle \equiv |0\rangle$

$$\langle P^4\rangle = \langle P_x^4 + P_y^4 + P_z^4 + 2P_x^2P_y^2 + 2P_x^2P_z^2 + 2P_y^2P_z^2\rangle = 3\langle P_x^4\rangle + 6(\langle P_x^2\rangle)^2$$

$$\Rightarrow E_0^{(1)} = \langle 0|H_1|0\rangle = \frac{-1}{8m^3c^2}\langle 0|P^4|0\rangle = \frac{-15\hbar^2\omega^2}{32mc^2}$$

2. $\psi_{nlm} = R_{nl}Y_l^m \qquad R_{nl}(\rho) = \rho^l \sum_{k=0}^{n-l-1} b_k \rho^k e^{-\rho/2} \qquad \rho = \frac{2r}{na_0}, \qquad b_{k+1} = \frac{k+l+1-n}{(k+1)(k+2l+2)}b_k$

for $l = n-1$, $\qquad R_{n(n-1)} = \rho^{n-1}\sum_{k=0}^{n-(n-1)-1} b_k \rho^k e^{-\rho/2} = \rho^{n-1} b_0 e^{-\rho/2} = C r^{n-1} e^{-r/na_0}$

$$\Rightarrow \psi_{n(n-1)m} = C r^{n-1} e^{-r/na_0}\, Y_{n-1}^m$$

$\langle \psi|\psi\rangle = 1 \quad \Rightarrow \quad 1 = C^2\int_0^\infty r^2 r^{2(n-1)} e^{-\frac{2r}{na_0}} dr = C^2 (2n)! \Big/ \left(\frac{2}{na_0}\right)^{2n+1}$

$\langle Y_l^m | Y_l^m\rangle = 1$

$\langle R_{nl}|R_{nl}\rangle = 1 \qquad \Rightarrow C = \left[\frac{2^{2n+1}}{(na_0)^{2n+1}(2n)!}\right]^{1/2}$

$$\Rightarrow \left\langle \frac{1}{r}\right\rangle_{n,n-1} = C^2\int_0^\infty \frac{1}{r} r^2 r^{2(n-1)} e^{-\frac{2r}{na_0}} dr = \frac{1}{a_0 n^2} \qquad \Rightarrow \left\langle\frac{1}{r}\right\rangle_{10} = \frac{1}{a_0}, \quad \left\langle\frac{1}{r}\right\rangle_{21} = \frac{1}{4a_0}, \quad \left\langle\frac{1}{r}\right\rangle_{32} = \frac{1}{9a_0}$$

$$\left\langle \frac{1}{r^2} \right\rangle_{n,n-1} = C^2 \int_0^\infty r^{2(n-1)} e^{-\frac{2r}{na_0}} dr = \frac{2}{(2n-1)n^3 a_0^2} = \frac{1}{(n-\frac{1}{2})n^3 a_0^2} = \frac{1}{a^2 n^3 (l+\frac{1}{2})} \Rightarrow \left\langle \frac{1}{r^2} \right\rangle_{10} = \frac{2}{a_0^2}, \left\langle \frac{1}{r^2} \right\rangle_{21} = \frac{1}{12a_0^2}, \left\langle \frac{1}{r^2} \right\rangle_{32} = \frac{2}{135a_0^2}$$

$$\left\langle \frac{1}{r^3} \right\rangle_{n,n-1} = C^2 \int_0^\infty \frac{1}{r^3} r^2 r^{2(n-1)} e^{-\frac{2r}{na_0}} dr = \frac{1}{a_0^3 n^4 (n-1)(n-\frac{1}{2})} = \frac{1}{a_0^3\, l(l+\frac{1}{2})(l+1) n^3}$$

$$\Rightarrow \left\langle \frac{1}{r^3} \right\rangle_{10} = \infty \qquad \left\langle \frac{1}{r^3} \right\rangle_{21} = \frac{1}{24a_0^3} \qquad \left\langle \frac{1}{r^3} \right\rangle_{32} = \frac{1}{405\, a_0^3}$$