1.

$$H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_2} \quad , \quad I_1 >> I_2$$

$$= \frac{L_x^2 + L_y^2 + L_z^2}{2I_1} + L_z^2 \left(\frac{1}{2I_2} - \frac{1}{2I_1} \right)$$

$$H = \frac{L^2}{2I_1} + L_z^2 \left(\frac{1}{2I_2} - \frac{1}{2I_1} \right)$$

This has eigenvalues $\frac{\ell(\ell+1)\hbar^2}{2I_1} + m^2\hbar^2 \left(\frac{1}{2I_2} - \frac{1}{2I_1} \right)$

or $\frac{\left[\ell(\ell+1) - m^2\right]\hbar^2}{2I_1} + \frac{m^2\hbar^2}{2I_2}$

In limit $I_1 >> I_2$, this is approx. $\frac{m^2\hbar^2}{2I_2}$.

2. Suppose $\langle L_x^2 \rangle = \langle L_y^2 \rangle = 0$

Then, $\langle L^2 \rangle = \langle L_z^2 \rangle$.

If $|\psi\rangle$ is an eigenstate of L^2, L_z, this implies

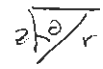
$\ell(\ell+1)\hbar^2 = m^2\hbar^2$ or $\ell(\ell+1) = m^2$

This can only be true if $\ell = 0$.

3.

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \quad (\text{constant})$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$


(geometry)

$$Y_{1\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\left(\cos\phi \pm i\sin\phi\right)$$

$$= \mp\sqrt{\frac{3}{8\pi}}\left(\sin\theta\cos\phi \pm i\sin\theta\sin\phi\right)$$

$$= \mp\sqrt{\frac{3}{8\pi}}\,(x \pm iy)$$

$$Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) = \sqrt{\frac{5}{16\pi}}\,\frac{3z^2 - r^2}{r^2}$$

$$Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}}\left(\sin\theta\cos\theta\right)e^{\pm i\phi}$$

$$= \mp\sqrt{\frac{15}{8\pi}}\cos\theta\left(\sin\theta\cos\phi \pm i\sin\theta\sin\phi\right)$$

$$= \mp\sqrt{\frac{15}{8\pi}}\,\frac{z(x \pm iy)}{r^2}$$

$$Y_{2\pm2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\left(\cos 2\phi \pm i\sin 2\phi\right)$$

$$= \sqrt{\frac{15}{32\pi}}\sin^2\theta\left(\cos^2\phi - \sin^2\phi \pm 2i\sin\phi\cos\phi\right)$$

$$= \sqrt{\frac{15}{32\pi}}\left[\left(\sin\theta\cos\phi\right)^2 - \left(\sin\theta\sin\phi\right)^2 \pm 2i\left(\sin\theta\sin\phi\right)\left(\sin\theta\cos\phi\right)\right]$$

$$= \sqrt{\frac{15}{32\pi}}\,\frac{(x \pm iy)^2}{r^2}$$

4.

$$\psi(x,y,z) = C(xy + yz + zx)\, e^{-\alpha r^2}$$

From problem 3,

$$xz = r^2\sqrt{\frac{2\pi}{15}}\left(Y_{2-1} - Y_{21}\right)$$

$$yz = i r^2\sqrt{\frac{2\pi}{15}}\left(Y_{21} + Y_{2-1}\right)$$

$$xy = -i r^2\sqrt{\frac{2\pi}{15}}\left(Y_{2-2} - Y_{22}\right)$$

$$\psi(r,\theta,\phi) = C\sqrt{\frac{2\pi}{15}}\left\{(1+i)Y_{2-1} - (1-i)Y_{21} - i\left(Y_{22} + Y_{2-2}\right)\right\} r^2 e^{-\alpha r^2}$$

Since all $Y_{\ell m}$ of form Y_{2m}, if a measurement of L^2 is made, the only possible eigenvalue is $2(2+1)\hbar^2 = 6\hbar^2$. $P(\ell=2) = \underline{1}$

If L_z is measured, possible eigenvalues are $\pm\hbar$, $\pm 2\hbar$.

$$\left.\begin{aligned} P(m=-2) &= P(m=2) = 1/6 \\ P(m=-1) &= P(m=1) = 1/3 \end{aligned}\right\} \quad 1/6 + 1/6 + 1/3 + 1/3 = 1$$

5.

$$V = \begin{cases} 0 & , \quad r > a \\ \infty & , \quad r < a \end{cases}$$

$$\langle \vec{x} | \psi^{(+)} \rangle = e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k}, \vec{k}')$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_{\ell} i^{\ell} (2\ell+1) A_\ell(r) P_\ell(\cos\theta)$$

$$A_\ell(r) = c_\ell^{(1)} h_\ell^{(1)}(r) + c_\ell^{(2)} h_\ell^{(2)}(r) \quad , \qquad \text{where}$$

$$h_\ell^{(1)}(r) = j_\ell + i n_\ell \qquad h_\ell^{(2)} = j_\ell - i n_\ell$$

are Hankel functions of the first and second kind and j_ℓ, n_ℓ are spherical bessel, neumann functions.

A bit of math (which I will leave out for now) gives a general expression

$$\tan\delta_\ell = \frac{ka\, j_\ell'(ka) - \beta_\ell j_\ell(ka)}{ka\, n_\ell'(ka) - \beta_\ell n_\ell(ka)} \quad , \qquad \text{where}$$

β_ℓ is the logarithmic derivative of A_ℓ determined by continuity at the boundary of the scattering region. For the special case of interest here, the wave function must vanish at the boundary. So, we have

$$j_\ell(ka)\cos\delta_\ell - n_\ell(ka)\sin\delta_\ell = 0$$

$$\tan\delta_\ell = \frac{j_\ell(ka)}{n_\ell(ka)}$$

$$\tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = \frac{\sin ka/ka}{-\cos ka/ka} = -\tan ka$$

$$\therefore \quad \delta_0 = -ka$$