HOMEWORK 5

1. $|\psi(t=0)\rangle = \frac{2}{\sqrt{7}}|u_4\rangle + i\frac{\sqrt{3}}{\sqrt{7}}|u_3\rangle$

$$|\psi(t)\rangle = \frac{2}{\sqrt{7}}|u_4\rangle e^{9i\omega t/2} + i\sqrt{\frac{3}{7}}|u_3\rangle e^{7i\omega t/2}$$

$$\hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\,(a - a^\dagger)$$

$$\langle\psi(t)|\hat{p}|\psi(t)\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\langle\psi(t)|\left(\frac{4}{\sqrt{7}}|u_3\rangle e^{9i\omega t/2} + -i2\sqrt{\frac{3}{7}}|u_4\rangle e^{7i\omega t/2}\right)$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}\left\{\langle\psi(t)|u_3\rangle\cdot\frac{4}{\sqrt{7}}e^{9i\omega t/2} - i2\sqrt{\frac{3}{7}}\langle\psi(t)|u_4\rangle e^{7i\omega t/2}\right\}$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}\left\{-i\sqrt{\frac{3}{7}}\cdot\frac{4}{\sqrt{7}}e^{i\omega t} - i2\sqrt{\frac{3}{7}}\,\frac{2}{\sqrt{7}}e^{-i\omega t}\right\}$$

$$= -\sqrt{\frac{m\hbar\omega}{2}}\cdot\frac{4\sqrt{3}}{7}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= -2\sqrt{\frac{3}{2}}\,\frac{\sqrt{m\hbar\omega}}{7}\cos(\omega t)$$

2. $\langle u_n | \hat{x} | u_n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle u_n | a + a^\dagger | u_n \rangle$

$= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \langle u_n | u_{n-1} \rangle \sqrt{n} + \sqrt{n+1} \langle u_n | u_{n+1} \rangle \right.$

$= 0$

$\langle u_n | \hat{p} | u_n \rangle = -i \sqrt{\frac{m\hbar\omega}{2}} \left\{ \langle u_n | a | u_n \rangle - \langle u_n | a^\dagger | u_n \rangle \right\}$

$= -i \sqrt{\frac{m\hbar\omega}{2}} \left\{ \sqrt{n} \langle u_n | u_{n-1} \rangle - \sqrt{n+1} \langle u_n | u_{n+1} \rangle \right\}$

$= 0$

$\langle u_n | x^2 | u_n \rangle = \frac{\hbar}{2m\omega} \langle u_n | aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger | u_n \rangle$

$= \frac{\hbar}{2m\omega} \left\{ \langle u_n | aa^\dagger | u_n \rangle + \langle u_n | a^\dagger a | u_n \rangle \right.$

$= \frac{\hbar}{2m\omega} \left\{ (n+1) + n \right\} \quad , n \geq 0$

$= \frac{\hbar}{m\omega} (n + \frac{1}{2})$

$\langle u_n | p^2 | u_n \rangle = -\frac{m\hbar\omega}{2} \langle u_n | aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger | u_n \rangle$

$= + \frac{m\hbar\omega}{2} \left\{ \langle u_n | aa^\dagger | u_n \rangle + \langle u_n | a^\dagger a | u_n \rangle \right\}$

$= m\hbar\omega (n + \frac{1}{2})$

$$\sqrt{\langle u_n|(x-\bar{x})^2|u_n\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$= \sqrt{\langle x^2\rangle}$$

$$\sigma_x = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$$

$$\sqrt{\langle u_n|(p-\bar{p})^2|u_n\rangle} = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$

$$= \sqrt{\langle p^2\rangle}$$

$$\sigma_p = \sqrt{m\hbar\omega\left(n+\frac{1}{2}\right)}$$

for u_0: $\sigma_x \sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\hbar\omega}{2}} = \frac{\hbar}{2}$

for u_1: $\sigma_x \sigma_p = \sqrt{\frac{3\hbar}{2m\omega}}\sqrt{\frac{3m\hbar\omega}{2}} = \frac{3\hbar}{2}$

4. Consider $|\psi\rangle, |\phi\rangle$, U a unitary operator.

If U unitary, $U^\dagger = U^{-1}$.

Now, define $|\tilde{\psi}\rangle = U|\psi\rangle$

$|\tilde{\phi}\rangle = U|\phi\rangle$.

Then,
$$\langle\tilde{\psi}|\tilde{\phi}\rangle = \langle\psi|U^\dagger U|\phi\rangle$$
$$= \langle\psi|\phi\rangle.$$

So, unitary operators preserve orthonormality.

Well, this only shows normality. Orthogonality follows if we consider a set of orthonormal basis vectors $\{|v_i\rangle\}$.

Then, let $|\tilde{v}_i\rangle = U|v_i\rangle$.

So,
$$\langle\tilde{v}_i|\tilde{v}_j\rangle = \langle v_i|U^\dagger U|v_j\rangle$$
$$= \langle v_i|v_j\rangle$$
$$= \delta_{ij}.$$

Well, if $\{|u_i\rangle\}$ complete,

$$\sum_i |u_i\rangle\langle u_i| = \mathbb{1}$$

$$U \sum_i |u_i\rangle\langle u_i| \, U^\dagger = U \mathbb{1} U^\dagger$$

$$\sum_i U|u_i\rangle\langle u_i| U^\dagger = U U^\dagger$$

$$\sum_i |\tilde{u}_i\rangle\langle \tilde{u}_i| = \mathbb{1}$$

5. If A hermitian, $A^\dagger = A$.

Then,

$$T = e^{iH} = \sum_j \frac{(iH)^j}{j!}$$

$$T^\dagger = \left(\sum_j \frac{(iH)^j}{j!}\right)^\dagger$$

$$= \sum_j \frac{(-iH^\dagger)^j}{j!}$$

$$= \sum_j (-1)^j \frac{(iH)^j}{j!}$$

$$= e^{-iH}$$

$$TT^\dagger = e^{iH} e^{-iH} = \underline{\mathbb{1}}$$

$$T^\dagger T = e^{-iH} e^{iH} = \underline{\mathbb{1}}$$

why? 0^{th} order: $(1)(1) = 1$

1^{st} order $(1 + iH)(1 - iH) = 1$ (to 1^{st} order)

$\vdots$

General result: $e^{A+B} = e^A e^B e^{-[A,B]/2}$

6. $$\langle u_i|\hat{x}|u_j\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle u_i|a+a^\dagger|u_j\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{j}\langle u_i|u_{j-1}\rangle + \sqrt{j+1}\langle u_i|u_{j+1}\rangle\right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{j}\,\delta_{i\,j-1} + \sqrt{j+1}\,\delta_{i\,j+1}\right\}$$

$$\langle u_i|\hat{p}|u_j\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\langle u_i|a-a^\dagger|u_j\rangle$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}\left\{\sqrt{j}\langle u_i|u_{j-1}\rangle - \sqrt{j+1}\langle u_i|u_{j+1}\rangle\right\}$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}\left\{\sqrt{j}\,\delta_{i\,j-1} - \sqrt{j+1}\,\delta_{i\,j+1}\right\}$$

7. $\langle u_i | xp | u_j \rangle = -\frac{i\hbar}{2} \langle u_i | aa + a^\dagger a - aa^\dagger - a^\dagger a^\dagger | u_j \rangle$

$$= -\frac{i\hbar}{2} \left\{ \langle u_i | aa | u_j \rangle + \langle u_i | a^\dagger a | u_j \rangle - \langle u_i | aa^\dagger | u_j \rangle \overset{\text{minus}}{+} \langle u_i | a^\dagger a^\dagger | u_j \rangle \right\}$$

$$= -\frac{i\hbar}{2} \left\{ \sqrt{j(j-1)} \langle u_i | u_{j-2} \rangle + j \langle u_i | u_j \rangle - (j+1) \langle u_i | u_j \rangle - \sqrt{(j+1)(j+2)} \langle u_i | u_{j+2} \rangle \right\}$$

$$= \frac{-i\hbar}{2} \left\{ \sqrt{j(j-1)}\, \delta_{i\,j-2} - \sqrt{(j+1)(j+2)}\, \delta_{i\,j+2} - \delta_{ij} \right\}$$

$$\sum_k \langle u_i | x | u_k \rangle \langle u_k | p | u_j \rangle$$

$$= \sum_k \left(-\frac{i\hbar}{2}\right) \left\{ \sqrt{k}\, \delta_{i\,k-1} + \sqrt{k+1}\, \delta_{i\,k+1} \right\} \times \left\{ \sqrt{j}\, \delta_{k\,j-1} - \sqrt{j+1}\, \delta_{k\,j+1} \right\}$$

$$= \left(-\frac{i\hbar}{2}\right) \sum_k \Big\{ \sqrt{kj}\, \delta_{i\,k-1} \delta_{k\,j-1} - \sqrt{k(j+1)}\, \delta_{i\,k-1} \delta_{k\,j+1} + \sqrt{j(k+1)}\, \delta_{i\,k+1} \delta_{k\,j-1} - \sqrt{(k+1)(j+1)}\, \delta_{i\,k+1} \delta_{k\,j+1}$$

But, for term like $\delta_{i\,k-1}\delta_{k\,j-1}$ we require

$$\left. \begin{array}{l} i = k-1 \\ k = j-1 \end{array} \right\} \quad i = j-2 \text{ or term is } 0$$

or

Thus,

$$= -\frac{i\hbar}{2}\left\{\sqrt{j(j-1)}\;\delta_{i,j-2} - \sqrt{(j+1)(j+2)}\;\delta_{i,j+2} - \delta_{ij}\right\}$$

8. $h(A^+)$ is a polynomial in A^+

$$h(A^+) = \sum_k b_k (A^+)^k$$

$$\begin{aligned}
Ah(A^+)\,u_0 &= A \sum_k b_k (A^+)^k u_0 \\
&= \sum_k b_k \, A (A^+)^k u_0 \\
&= \sum_k b_k \sqrt{k!}\, A u_k \\
&= \sum_k k b_k \sqrt{(k-1)!}\; u_{k-1} \\
&= \sum_k k b_k (A^+)^{k-1} u_0 \\
&= \frac{dh(A^+)}{dA^+} u_0
\end{aligned}$$

9. $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \left(|u_0\rangle + |u_1\rangle \right)$

Schroedinger!

$$\langle \psi(t) | \hat{x} | \psi(t) \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle \psi(t) | (a + a^\dagger) | \psi(t) \rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|u_0\rangle e^{i\omega t/2} + |u_1\rangle e^{3i\omega t/2} \right)$$

$$= \frac{1}{\sqrt{2}} \sqrt{\frac{\hbar}{2m\omega}} \langle \psi(t) | \left(|u_0\rangle e^{3i\omega t/2} + |u_1\rangle e^{i\omega t/2} + \sqrt{2}\, |u_2\rangle e^{i\omega t/2} \right)$$

$$= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \left\{ \langle u_0 | u_0 \rangle e^{i\omega t} + \langle u_1 | u_1 \rangle e^{-i\omega t} \right\}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)$$

Heisenberg:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_0\rangle + |\psi_1\rangle\right)$$

NO TIME DEPENDENCE

$$\hat{x} \longrightarrow U^{+}\hat{x}\,U$$

$$\langle\psi|U^{+}\hat{x}U\psi\rangle = \langle\psi U|\hat{x}|U\psi\rangle$$

$$= \langle\psi(t)|\hat{x}|\psi(t)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$$