Homework 3

1. $[p, x^n]g = \left[\frac{\hbar}{i}\frac{\partial}{\partial x}, x^n\right] g$

test function

$$= \frac{\hbar}{i}\frac{\partial}{\partial x}(x^n g) - x^n \frac{\hbar}{i}\frac{\partial g}{\partial x}$$

$$= \frac{\hbar}{i}\frac{\partial x^n}{\partial x} g + \cancel{\frac{\hbar}{i} x^n \frac{\partial g}{\partial x}} - \cancel{x^n \frac{\hbar}{i}\frac{\partial g}{\partial x}}$$

$$= \frac{\hbar}{i} n x^{n-1} g$$

So, $[p, x^n] = -i\hbar n x^{n-1}$.

In general, $[p, f(x)] = -i\hbar \frac{\partial f}{\partial x}$.

Now, consider $[p^m, x^n]$.

(1) $[p, x^n] = -i\hbar n x^{n-1} = px^n$

(2) $[p^2, x^n] = p[p, x^n] + [p, x^n]p$

$$= p^2 x^n + px^n p$$

(3) $[p^3, x^n] = p[p^2, x^n] + [p, x^n]p^2$

$$= p(p^2 x^n + px^n p) + (px^n)p^2$$

$$= p^3 x^n + p^2 x^n p + px^n p^2$$

(4) $[p^4, x^n] = p[p^3, x^n] + [p, x^n]p^3$

$$= p(p^3 x^n + p^2 x^n p + px^n p^2) + px^n p^3$$

$$= p^4 x^n + p^3 x^n p + p^2 x^n p^2 + px^n p^3$$

So, $[p^m, x^n] = \sum_{k=0}^{m-1} p^{m-k} x^n p^k$

2. $P\psi(x) = \psi(-x)$

$P^2\psi(x) = P(P\psi(x)) = P\psi(-x) = \psi(x)$

+1 So, P has eigenvalues ± 1.

Is P Hermitian?

Need to show that $\int dx\, \psi^*(x) P \psi(x) = \left\{ \int dx\, \psi^* P^\dagger \psi \right\}^*$

$$\int dx\, \psi^*(x) P \psi(x) = \int dx\, \psi^*(x) \psi(-x)$$

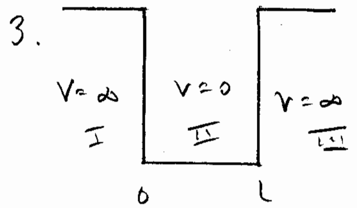
$$= \left\{ \int dx\, \psi^*(-x)\, \psi(x) \right\}^*$$

$$= \left\{ \int dx\, (P\psi(x))^\dagger \psi(x) \right\}^\dagger$$

$$= \left\{ \int dx\, \psi^*(x) P^\dagger \psi(x) \right\}^*$$

+1 $\therefore$ P Hermitian.

3.



In regions I, III, $\psi = 0$

In region II, $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad , \quad k^2 = \frac{2mE}{\hbar^2}$$

So, $\psi(x) = A\sin kx + B\cos kx$

What are our boundary conditions? (1) $\psi_I(0) = \psi_{II}(0) = 0$

(2) $\psi_{II}(L) = \psi_{III}(L) = 0$

So, (1) gives $0 = B$.

Thus, $\psi(x) = A\sin kx$

(2) gives $0 = \sin kL \longrightarrow kL = n\pi$

$k = \frac{n}{L}\pi$

So, $\psi_n(x) = A\sin\left(\frac{n\pi}{L}x\right)$.

→ Normalization determines A.

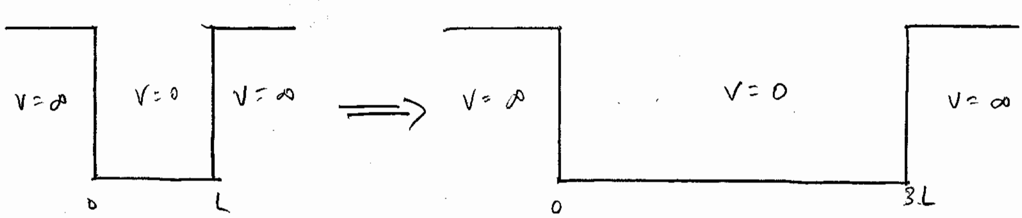
$$1 = |A|^2\int_0^L dx\,\sin^2\left(\frac{n\pi}{L}x\right) = |A|^2\left(\frac{L}{2}\right)$$

$$\therefore\ A = \sqrt{\frac{2}{L}}$$

$$\psi_n^{(L)}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

In the 1^{st} excited state, $\psi_2^{(L)}(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi}{L}x\right)$.

Now, suppose we suddenly move the wall at $x = L$ to $x = 3L$.



Our new wave functions are found by making the replacement $L \to 3L$.

So, $\psi_n^{(3L)}(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{n\pi}{3L} x\right)$.

In this new configuration, $\psi_1^{(3L)}(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{\pi}{3L} x\right)$ (ground state)

$\psi_2^{(3L)}(x) = \sqrt{\frac{2}{3L}} \sin\left(\frac{2\pi}{3L} x\right)$ (1^{st} excited state)

(1) probability to be in 3L ground state

$$P = |\langle \psi_1^{(3L)} | \psi_2^{(L)} \rangle|^2$$

$$= \left| \frac{1}{\sqrt{3}} \frac{2}{L} \int_0^L dx \ \sin\left(\frac{2\pi}{L} x\right) \sin\left(\frac{\pi}{3L} x\right) \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \int_0^L dx \left\{ \cos\left[\left(\frac{2\pi}{L} - \frac{\pi}{3L}\right)x\right] - \cos\left[\left(\frac{2\pi}{L} + \frac{\pi}{3L}\right)x\right] \right\} \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \int_0^L dx \left\{ \cos\left(\frac{5\pi}{3L} x\right) - \cos\left(\frac{7\pi}{3L} x\right) \right\} \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \left\{ \frac{3L}{5\pi} \sin\left(\frac{5\pi}{3L} x\right) - \frac{3L}{7\pi} \sin\left(\frac{7\pi}{3L} x\right) \right\} \Big|_0^L \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \left\{ \frac{3L}{5\pi} \sin\left(\frac{5\pi}{3}\right) - \frac{3L}{7\pi} \sin\left(\frac{7\pi}{3}\right) \right\} \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left\{ -\frac{1}{5} \frac{\sqrt{3}}{2} - \frac{1}{7} \frac{\sqrt{3}}{2} \right\} \right|^2$$

$$= \left| \frac{-3}{2\pi} \left(\frac{1}{5} + \frac{1}{7} \right) \right|^2$$

$$= \left(\frac{18}{35\pi} \right)^2$$

$$= \frac{324}{1225\pi^2} \approx 0.027$$

(2) probability to be in 3L 1^{st} excited state

$$P = \left| \langle \psi_2^{(3L)} | \psi_2^{(L)} \rangle \right|^2$$

$$= \left| \frac{2}{\sqrt{3}L} \int_0^L dx \, \sin\left(\frac{2\pi}{L} x\right) \sin\left(\frac{2\pi}{3L} x\right) \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \int_0^L dx \left\{ \cos\left(\frac{4\pi}{3L} x\right) - \cos\left(\frac{8\pi}{3L} x\right) \right\} \right|^2$$

$$= \left| \frac{1}{L\sqrt{3}} \left\{ \frac{3L}{4\pi} \sin\left(\frac{4\pi}{3L} x\right) - \frac{3L}{8\pi} \sin\left(\frac{8\pi}{3L} x\right) \right\} \Big|_0^L \right|^2$$

$$= \left| \frac{\sqrt{3}}{\pi} \left\{ \frac{1}{4} \sin\left(\frac{4\pi}{3}\right) - \frac{1}{8} \sin\left(\frac{8\pi}{3}\right) \right\} \right|^2$$

$$= \left| \frac{\sqrt{3}}{4\pi} \left\{ -\frac{\sqrt{3}}{2} - \frac{1}{2} \frac{\sqrt{3}}{2} \right\} \right|^2$$

$$= \left| -\frac{3}{8\pi} \cdot \frac{3}{2} \right|^2$$

$$= \left(\frac{9}{16\pi} \right)^2$$

$$= \frac{81}{256\pi^2} \approx 0.032$$

4 $L = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$, $\psi(\theta)$ on $-\pi < \theta < \pi$

$\psi(-\pi) = \psi(\pi)$

→ Show L is Hermitian

That is, $\int d\theta\, \psi^* L \psi = \left\{ \int d\theta\, \psi^* L^\dagger \psi \right\}^*$

(This shows, in fact, that L has real expectation value: $\langle L\rangle - \langle L^\dagger\rangle^* = 0$ if $L = L^\dagger$)

→ let's first check derivative operator $\partial/\partial\theta$

$$\int d\theta\, \psi^* \frac{\partial}{\partial\theta}\psi = \int d\theta \left\{ \frac{\partial}{\partial\theta}(\psi^*\psi) - \frac{\partial\psi^*}{\partial\theta}\psi \right\}$$

$$= \cancel{\psi^*\psi \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} d\theta\, \frac{\partial\psi^*}{\partial\theta}\psi$$

0 since $\psi(-\pi) = \psi(\pi)$

$$= -\int d\theta\, \psi \frac{\partial}{\partial\theta}\psi^*$$

$$= -\left\{ \int d\theta\, \psi^* \left(\frac{\partial}{\partial\theta}\right)^\dagger \psi \right\}^*$$

So, $\left(\frac{\partial}{\partial\theta}\right)^\dagger = -\frac{\partial}{\partial\theta}$ → $\frac{\partial}{\partial\theta}$ anti-hermitian.

Thus, $L = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$ gives $L^\dagger = \left(\frac{\hbar}{i}\right)^* \left(\frac{\partial}{\partial\theta}\right)^\dagger$

$$= -\frac{\hbar}{i}\left(-\frac{\partial}{\partial\theta}\right)$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial\theta} = L.$$

∴ L Hermitian.

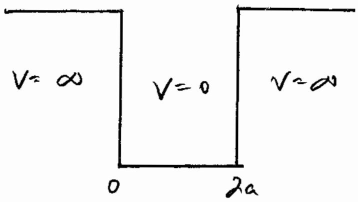
So, $\langle L \rangle - \langle L \rangle^* = \int_{-\pi}^{\pi} d\theta \left\{ \psi^* \left(\frac{\hbar}{i}\right) \frac{\partial}{\partial \theta} \psi - \psi \left(-\frac{\hbar}{i}\right) \frac{\partial \psi^*}{\partial \theta} \right\}$

$$= \frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta \left\{ \psi^* \frac{\partial \psi}{\partial \theta} + \psi \frac{\partial \psi^*}{\partial \theta} \right\}$$

$$= \frac{\hbar}{i} \int_{-\pi}^{\pi} d\theta \, \frac{\partial}{\partial \theta} \left(\psi^* \psi \right) = 0$$

Thus, L has real expectation values.

5.



From problem 3 we found,

$$\psi_n = \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi}{2a}x\right)$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^{2a} dx \; e^{-ipx/\hbar} \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi}{2a}x\right)$$

$$\sin\left(\frac{n\pi}{2a}x\right) = \frac{1}{2i}\left(e^{i\frac{n\pi}{2a}x} - e^{-i\frac{n\pi}{2a}x}\right)$$

$$\phi(p) = \frac{-i}{\sqrt{8\pi\hbar a}} \int_0^{2a} dx \left\{ e^{-i(p/\hbar - \frac{n\pi}{2a})x} - e^{-i(\frac{p}{\hbar} + \frac{n\pi}{2a}\hbar)} \right\}$$

$$= -\frac{i}{\sqrt{8\pi\hbar a}} \left\{ \frac{1}{-i} \frac{e^{-i(p/\hbar - \frac{n\pi}{2a})x}}{p/\hbar - \frac{n\pi}{2a}} - \frac{1}{-i} \frac{e^{-i(p/\hbar + \frac{n\pi}{2a})x}}{p/\hbar + \frac{n\pi}{2a}} \right\}\Bigg|_0^{2a}$$

$$= + \frac{1}{\sqrt{8\pi\hbar a}} \left\{ \frac{e^{-i(\frac{2pa}{\hbar} - n\pi)} - 1}{p/\hbar - \frac{n\pi}{2a}} - \frac{e^{-i(\frac{2pa}{\hbar} + n\pi)} - 1}{p/\hbar + \frac{n\pi}{2a}} \right\}$$

$$= \frac{1}{\sqrt{8\pi\hbar a}} \left\{ \frac{\left(\frac{p}{\hbar} + \frac{n\pi}{2a}\right) e^{-i(\frac{2pa}{\hbar} - n\pi)} - \left(\frac{p}{\hbar} - \frac{n\pi}{2a}\right) e^{-i(\frac{2pa}{\hbar} + n\pi)} - \left(\frac{p}{\hbar} + \frac{n\pi}{2a}\right) + \left(\frac{p}{\hbar} - \frac{n\pi}{2a}\right)}{(p/\hbar)^2 - \left(\frac{n\pi}{2a}\right)^2} \right.$$

$$e^{\pm in\pi} = \cos(\pm n\pi) + i\cancel{\sin(\pm n\pi)}^{\,0} = \cos n\pi = (-1)^n$$

$$\phi(p) = \frac{1}{\sqrt{8\pi\hbar a}} \frac{1}{\left(\frac{p}{\hbar}\right)^2 - \left(\frac{n\pi}{2a}\right)^2} \left\{ \cancel{\frac{p}{\hbar}(-1)^n \left(e^{-i\frac{2pa}{\hbar}} - e^{-i\frac{2pa}{\hbar}}\right)}^{\,0} + \frac{n\pi}{2a}(-1)^n \left(e^{-i\frac{2pa}{\hbar}} + e^{-i\frac{2pa}{\hbar}}\right) - \frac{n\pi}{a} \right.$$

$$= \frac{\frac{n\pi}{a}}{\sqrt{8\pi\hbar a}} \frac{e^{-ipa/\hbar}}{\left(\frac{p}{\hbar}\right)^2 - \left(\frac{n\pi}{2a}\right)^2} \left\{ (-1)^n e^{-ipa/\hbar} - e^{ipa/\hbar} \right\}$$

$$= \frac{\frac{n\pi}{a}}{\sqrt{8\pi\hbar a}} \frac{e^{-ipa/\hbar}}{\left(\frac{n\pi}{2a}\right)^2 - \left(\frac{p}{\hbar}\right)^2} \left\{ e^{ipa/\hbar} - (-1)^n e^{-ipa/\hbar} \right\}$$

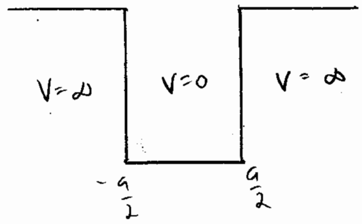
$$= \frac{\frac{2n\pi}{a}}{\sqrt{8\pi\hbar a}} \frac{e^{-ipa/\hbar}}{\left(\frac{n\pi}{2a}\right)^2 - \left(\frac{p}{\hbar}\right)^2} \begin{cases} \cos(pa/\hbar), & n \text{ odd} \\ \sin(pa/\hbar), & n \text{ even} \end{cases}$$

So, the probability to find the particle w/ momentum between p and $p+dp$ is $|\phi(p)|^2 \, dp$.

$$\text{So,} \quad |\phi(p)|^2 dp = \left(\frac{2n\pi}{a}\right)^2 \left(\frac{1}{8\pi\hbar a}\right) \frac{1}{\left[\left(\frac{n\pi}{2a}\right)^2 - \left(\frac{p}{\hbar}\right)^2\right]^2} \begin{cases} \cos^2(pa/\hbar), & n \text{ odd} \\ \sin^2(pa/\hbar), & n \text{ even} \end{cases}$$

Since a free particle of momentum p has energy $E_p = \frac{p^2}{2m}$, which need not equal the energy of a state of the infinite square well, energy is not conserved.

6.



$$\psi(x,0) = \sqrt{\frac{2}{L}}, \quad -\frac{L}{2} < x < 0$$
$$= 0, \quad \text{elsewhere}$$

For an infinite square well on $\left(-\frac{L}{2}, \frac{L}{2}\right)$, we have

$$\psi(x) = A\cos(kx) + B\sin(kx), \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

Since potential is even, we can write ψ as even and odd functions.

Thus,

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi}{L}x\right), & n \text{ odd} \\ \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right), & n \text{ even} \end{cases}$$

Now, let's express $\psi(x,0)$ in terms of $\psi_n(x)$.

$$\psi(x,0) = \sum_n c_n \psi_n(x) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} a_n \cos\left[\frac{(2n-1)\pi}{L}x\right] + b_n \sin\left(\frac{2n\pi}{L}x\right)$$

Using Fouriers trick

$$\int_{-L/2}^{L/2} dx\, \psi(x,0)\cos\left(\frac{(2n'+1)\pi}{L}x\right) = \sqrt{\frac{2}{L}}\, a_n \int_{-L/2}^{L/2} dx\, \underbrace{\cos\left(\frac{(2n+1)\pi}{L}x\right)\cos\left(\frac{(2n'+1)\pi}{L}x\right)}_{\text{only} \neq 0 \text{ for } n=n' \text{ (orthogonality)}}$$

$$\cancel{\sqrt{\frac{2}{L}}} \int_{-L/2}^{0} dx\, \cos\left(\frac{(2n+1)\pi}{L}x\right) = \cancel{\sqrt{\frac{2}{L}}}\, a_n \int_{-L/2} dx\, \cos^2\left(\frac{(2n-1)\pi}{L}x\right)$$

$$\frac{L}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi}{2}\right) = a_n\left(\frac{L}{2}\right)$$

$$\boxed{a_n = \frac{2}{(2n+1)\pi}(-1)^n}$$

Similarly

$$\int_{-L/2}^{0} dx\, \sin\left(\frac{2n\pi}{L}x\right) = b_n \int_{-L/2}^{L/2} dx\, \sin^2\left(\frac{(2n\pi)}{L}x\right)$$

$$-\frac{L}{2n\pi}\left(1-(-1)^n\right) = b_n\left(\frac{L}{2}\right) \implies b_n = -\frac{1}{n\pi}\left(1-(-1)^n\right)$$

So, $\psi(x,0) = \sqrt{\frac{2}{L}} = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(2n-1)\pi} \cos\left[\frac{(2n-1)\pi}{L} x\right]$

$- \frac{1}{n\pi}\left(1-(-1)^n\right) \sin\left(\frac{2n\pi}{L} x\right)$

$$\psi(x,t) = \psi(x,0)\, e^{-iEt/\hbar}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$\psi(x,t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(2n-1)} \cos\left[\frac{(2n-1)\pi}{L} x\right] e^{-i\frac{(2n-1)^2\pi^2\hbar}{2mL^2} t}$$

$$- \frac{1}{n\pi}\left(1-(-1)^n\right) \sin\left(\frac{2n\pi}{L} x\right) e^{-i\frac{2n^2\pi^2\hbar}{mL^2} t}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx\, e^{-ipx/\hbar}\, \psi(x,0)$$

$$= \frac{1}{\sqrt{L\pi\hbar}} \int_{-L/2}^{0} dx\, e^{-ipx/\hbar}$$

$$= \frac{1}{\sqrt{L\pi\hbar}} \frac{\hbar}{-ip}\left(1 - e^{ipL/2\hbar}\right) = \sqrt{\frac{\hbar}{\pi L}}\, \frac{2}{p}\, e^{ipL/4\hbar} \sin\left(\frac{pL}{4\hbar}\right)$$

$\rightarrow$ $\phi(p)$ is explicitly time-independent. It cannot be used to calculate $\psi(x,t)$.