9/21/06

HOMEWORK 2

1. $\phi(p) = \delta(5p - \hbar k_0)$ What is $\psi(x)$?

$$\psi(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dp\ \phi(p)\, e^{ipx/\hbar}$$

$$= \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dp\ \delta(5p - \hbar k_0)\, e^{ipx/\hbar}$$

Now, $\delta(5p - \hbar k_0) = \delta\left(5\left(p - \frac{\hbar k_0}{5}\right)\right)$

$$= \frac{1}{5}\delta\left(p - \frac{1}{5}\hbar k_0\right)$$

So, $\psi(x) = \dfrac{1/5}{(2\pi\hbar)^{1/2}} \displaystyle\int_{-\infty}^{\infty} dp\ \delta\left(p - \frac{1}{5}\hbar k_0\right) e^{ipx/\hbar}$

$$= \frac{1/5}{(2\pi\hbar)^{1/2}}\, e^{ik_0 x/5}$$

Check:

$$\phi(p) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx\ \psi(x)\, e^{-ipx/\hbar}$$

$$= \frac{1/5}{(2\pi\hbar)} \int_{-\infty}^{\infty} dx\ e^{-i(p - \frac{1}{5}\hbar k_0)x/\hbar}$$

$$= \frac{1}{5}\, \frac{1}{\cancel{(2\pi\hbar)}}\, \cancel{(2\pi\hbar)}\ \delta\left(p - \frac{1}{5}\hbar k_0\right)$$

$$= \frac{1}{5}\delta\left(p - \frac{1}{5}\hbar k_0\right)$$

$$= \delta(5p - \hbar k_0)\ \checkmark$$

(2) Prove $\int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x)\, dx = \int_{-\infty}^{\infty} \phi^*(p) \left(i\hbar \frac{\partial}{\partial p} \right) \phi(p)\, dp.$

Proof: (Fourier transform)

$$\int_{-\infty}^{\infty} \psi^*(x)\, x\, \psi(x)\, dx$$

$$= \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dp\, \phi(p)\, e^{ipx/\hbar}$$

$$= \int_{-\infty}^{\infty} dp\, \phi(p)\, x\, \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx\, \psi^*(x)\, e^{ipx/\hbar}$$

$$= \int_{-\infty}^{\infty} dp\, \phi(p) \left(\frac{\hbar}{i} \frac{\partial}{\partial p} \right) \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx\, \psi^*(x)\, e^{ipx/\hbar}$$

$$= \int_{-\infty}^{\infty} dp\, \phi(p) \left(\frac{\hbar}{i} \frac{\partial}{\partial p} \right) \phi^*(p)$$

Proof (Dirac Notation)

$$\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle$$

$$= \int dp\, dp' \, \langle \psi | p \rangle \langle p | \hat{x} | p' \rangle \langle p' | \psi \rangle$$

$$= \int dp\, dp'\, dx\, dx' \, \langle \psi | p \rangle \langle p | x \rangle \langle x | \hat{x} | x' \rangle \langle x' | p' \rangle \langle p' | \psi \rangle$$

Now, $\hat{x} | x \rangle = x | x \rangle$.

So, $\langle x | \hat{x} | x' \rangle = \langle x | x' | x' \rangle$

$$= x' \langle x | x' \rangle$$

$$= x' \, \delta(x - x')$$

$$= \int dp\, dp'\, dx\, dx' \, \langle \psi | p \rangle \langle p | x \rangle \, x' \, \delta(x - x') \, \langle x' | p' \rangle \langle p' | \psi \rangle$$

$$= \int dp\, dp'\, dx \, \langle \psi | p \rangle \langle p | x \rangle \, x \, \langle x | p' \rangle \langle p' | \psi \rangle$$

$$= \int dp\, dp'\, dx \, \langle \psi | p \rangle \, e^{-ipx/\hbar} \, x \, e^{ip'x/\hbar} \langle p' | \psi \rangle$$

$$= \int dp\, dp'\, dx \, \langle \psi | p \rangle \, e^{-ipx/\hbar} \left(\frac{\hbar}{i} \frac{\partial}{\partial p'} \right) e^{ip'x/\hbar} \langle p' | \psi \rangle$$

$$= \int dp\, dp'\, dx \, \langle \psi | p \rangle \langle p | x \rangle \langle x | p' \rangle \left(\frac{\hbar}{i} \frac{\partial}{\partial p'} \right) \langle p' | \psi \rangle$$

$$= \int dp\, dp' \, \langle \psi | p \rangle \left(\frac{\hbar}{i} \frac{\partial}{\partial p'} \right) \langle p' | \psi \rangle \, \delta(p - p')$$

$$= \int dp \, \langle \psi | p \rangle \left(\frac{\hbar}{i} \frac{\partial}{\partial p} \right) \langle p | \psi \rangle$$

$$= \int dp \, \psi^*(p) \left(\frac{\hbar}{i} \frac{\partial}{\partial p} \right) \psi(p)$$

(3) $\psi(x) = Ce^{-x^2/\alpha}$.

Compute $\Delta x = \sqrt{\langle\psi|(x-\langle x\rangle)^2|\psi\rangle}$

$$(x-\langle x\rangle)^2 = x^2 - 2x\langle x\rangle + \langle x\rangle^2$$

$$\langle\psi|(x-\langle x\rangle)^2|\psi\rangle = \langle\psi|(x^2 - 2x\langle x\rangle + \langle x\rangle^2|\psi\rangle$$

$$= \langle x^2\rangle - 2\langle x\rangle^2 + \langle x\rangle^2$$

$$= \langle x^2\rangle - \langle x\rangle^2$$

→ Normalize $\psi(x)$

$$\langle\psi|\psi\rangle = |c|^2 \int_{-\infty}^{\infty} dx\ e^{-2x^2/\alpha} =$$

what is $\int_{-\infty}^{\infty} e^{-\beta x^2} dx$.

$$\int_{-\infty}^{\infty} e^{-\beta x^2} dx \int_{-\infty}^{\infty} e^{-\beta y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\ e^{-\beta(x^2+y^2)}$$

$$= \int_0^{2\pi}\int_0^{\infty} r\,dr\,d\theta\ e^{-\beta r^2}$$

$$= 2\pi \int_0^{\infty} dr\ r e^{-\beta r^2}$$

$$= -\frac{2\pi}{2\beta} e^{-\beta r^2}\Big|_0^{\infty}$$

$$= \frac{\pi}{\beta}$$

$$\therefore \int_{-\infty}^{\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}$$

So, $\int_{-\infty}^{\infty} e^{-2x^2/\alpha}\, dx = \sqrt{\frac{\alpha\pi}{2}}$ $\quad (\beta = 2/\alpha)$.

So, $|c|^2 \left(\sqrt{\frac{\alpha\pi}{2}}\right) = 1$.

$$\therefore \quad c = \left(\frac{2}{\pi\alpha}\right)^{1/4}$$

Thus, $\psi(x) = \left(\frac{2}{\pi\alpha}\right)^{1/4} e^{-x^2/\alpha}$.

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \psi(x)$$

$$= \left(\frac{2}{\pi\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dx \;\; x\, e^{-2x^2/\alpha}$$

$$= 0 \qquad (\text{rotational symmetry})$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x^2\, \psi(x)$$

$$= \left(\frac{2}{\pi\alpha}\right)^{1/2} \int_{-\infty}^{\infty} dx\, x^2 e^{-2x^2/\alpha}$$

Ok, lets take a math side-trip.

What is $I = \int_{-\infty}^{\infty} dx\; x^{2n} e^{-\beta x^2}$? (only need to consider x since odd powers have $I=$

Well, $\int_{-\infty}^{\infty} dx\; e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}$.

Now, $$\int_{-\infty}^{\infty} dx\; x^{2n} e^{-\beta x^2} = \int_{-\infty}^{\infty} dx\; \left(-\frac{\partial}{\partial\beta}\right)^n e^{-\beta x^2}$$

$$= \left(-\frac{\partial}{\partial\beta}\right)^n \int_{-\infty}^{\infty} dx\; e^{-\beta x^2}$$

$$= \left(-\frac{\partial}{\partial\beta}\right)^n \left(\sqrt{\frac{\pi}{\beta}}\right)$$

$$= \frac{\sqrt{\pi}}{\beta^{(n+\frac{1}{2})}} \frac{(2n-1)!!}{2^n}, \quad n \geq 1$$

So, $$\int_{-\infty}^{\infty} dx\; x^2 e^{-2x^2/\alpha} = \frac{\sqrt{\pi}}{(2/\alpha)^{3/2}} \frac{1}{2} = \frac{\alpha\sqrt{\pi\alpha}}{2^{5/2}}$$

Thus, $$\langle x^2 \rangle = \left(\frac{2}{\pi\alpha}\right)^{1/2} \frac{1\alpha\sqrt{\pi\alpha}}{2^{5/2}} = \frac{1}{4}\alpha$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{\sqrt{\alpha}}{2}$$

(4) $\langle p^n \rangle$, $\psi(x) = \left(\frac{2}{\pi\alpha}\right)^{1/4} e^{-x^2/\alpha}$

$$\phi(p) = \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx \; \psi(x) \; e^{-ipx/\hbar}$$

$$= \frac{1}{(2\pi\hbar)^{1/2}} \left(\frac{2}{\pi\alpha}\right)^{1/4} \int_{-\infty}^{\infty} dx \; e^{-x^2/\alpha} \; e^{-ipx/\hbar}$$

$$e^{-\left(\frac{x^2}{\alpha} + ip\frac{x}{\hbar}\right)} = e^{-\frac{1}{\alpha}\left(x^2 + ip\alpha\frac{x}{\hbar} - \frac{p^2\alpha^2}{4\hbar^2} + \frac{p^2\alpha^2}{4\hbar^2}\right)}$$

$$= e^{-\frac{1}{\alpha}\left(x + i\frac{p\alpha}{2\hbar}\right)^2} e^{-\frac{p^2\alpha}{4\hbar^2}}$$

$$\phi(p) = \frac{1}{(2\pi\hbar)^{1/2}} \left(\frac{2}{\pi\alpha}\right)^{1/4} e^{-\frac{p^2\alpha}{4\hbar^2}} \underbrace{\int_{-\infty}^{\infty} dx \; e^{-\left(x + i\frac{p\alpha}{2\hbar}\right)^2/\alpha}}_{= I}$$

Now I can be treated as a gaussian and thus

$$\phi(p) = \frac{(2\pi\alpha)^{1/4}}{(2\pi\hbar)^{1/2}} e^{-\frac{p^2\alpha}{4\hbar^2}} = \phi^*(p)$$

So, $\langle p^n \rangle = \int_{-\infty}^{\infty} dp \; \phi^*(p) \; p^n \; \phi(p)$

$$= \frac{(2\pi\alpha)^{1/2}}{2\pi\hbar} \underbrace{\int_{-\infty}^{\infty} dp \; p^n \; e^{-\frac{\alpha}{2\hbar^2}p^2}}_{= I(n)}$$

For odd n, $n = 2k+1$, $k \in \mathbb{Z}$,

$$I(2k+1) = \int_{-\infty}^{\infty} dp \; p^{2k+1} e^{-\frac{\alpha}{2\hbar^2}p^2}$$

$$= 0 \quad (\text{rotational symmetry})$$

Thus, only even terms remain

$$I(2n) = \int_{-\infty}^{\infty} dp \; p^{2n} e^{-\frac{\alpha}{2\hbar^2}p^2} = \sqrt{\frac{2\pi\hbar^2}{\alpha}}, \quad n=0$$

$$= \frac{\sqrt{\pi}\,(2n+1)!!}{2^n}\left(\frac{2\hbar^2}{\alpha}\right)^{(n+1/2)}$$

So, $\langle p^{2n}\rangle = \frac{(2\pi\alpha)^{1/2}}{2\pi\hbar}\frac{\sqrt{\pi}\,(2n-1)!!}{2^n}\left(\frac{2\hbar^2}{\alpha}\right)^{(n+1/2)}$,

Thus, $\langle p\rangle = 0$ and $\langle p^2\rangle = \frac{(2\pi\alpha)^{1/2}}{2\pi\hbar}\frac{\sqrt{\pi}\cdot 1}{2}\cdot\left(\frac{2\hbar^2}{\alpha}\right)^{3/2}$

$$= \frac{4\pi\sqrt{\alpha}}{4\pi\hbar}(1)\frac{\hbar^2}{\alpha^{3/2}}$$

$$= \frac{\hbar^2}{\alpha}$$

So, $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar}{\sqrt{\alpha}}$

Using result from previous problem, $\Delta x \Delta p = \left(\frac{\sqrt{\alpha}}{2}\right)\left(\frac{\hbar}{\sqrt{\alpha}}\right) = \frac{\hbar}{2}$

(5)

$$\psi(x,t) = \frac{\sqrt{\alpha}}{(2\pi)^{3/4}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{4}(k-k_0)^2} e^{i[kx - \omega(k)t]} \, dk,$$

where $\omega(k) = \frac{\hbar k^2}{2m}$ is the dispersion relation.

$$\omega(k) = \frac{\hbar k_0^2}{2m} + \frac{\hbar k_0}{m}(k-k_0) + \frac{\hbar}{2m}(k-k_0)^2$$

So, $-\frac{\alpha^2}{4}(k-k_0)^2 + ikx - i\omega(k)t$

$$= -\frac{\alpha^2}{4}(k-k_0)^2 + i(k-k_0)x + ik_0x - it\left[\frac{\hbar k_0^2}{2m} + \frac{\hbar k_0}{m}(k-k_0) + \frac{\hbar}{2m}(k-k_0)^2\right.$$

$$= \left(-\frac{\alpha^2}{4} - \frac{i\hbar t}{2m}\right)(k-k_0)^2 + \left(ix - \frac{i\hbar k_0 t}{m}\right)(k-k_0) + \left[+ik_0x - \frac{i\hbar k_0^2 t}{2m}\right]$$

$$= -\frac{\alpha^2}{4}\left(1 + \frac{2i\hbar t}{m\alpha^2}\right)(k-k_0)^2 - \frac{\alpha^2}{4}\left(-\frac{4ix}{\alpha^2} + \frac{4i\hbar k_0 t}{m\alpha^2}\right)(k-k_0) - \frac{\alpha^2}{4}\left[-\frac{4ik_0x}{\alpha^2} + \frac{2i\hbar k_0^2 t}{m\alpha^2}\right]$$

$$= -\frac{\alpha^2}{4}\left(1 + \frac{2i\hbar t}{m\alpha^2}\right)\left\{(k-k_0)^2 + \frac{\frac{4i\hbar k_0 t}{m\alpha^2} - \frac{4ix}{\alpha^2}}{1 + \frac{2i\hbar t}{m\alpha^2}}(k-k_0) + \frac{\frac{2i\hbar k_0 t}{m\alpha^2} - \frac{4ik_0 x}{\alpha^2}}{1 + \frac{2i\hbar t}{m\alpha^2}}\right\}$$

$$= -\frac{\alpha^2}{4}\left(1 + \frac{2i\hbar t}{m\alpha^2}\right)\left[k - k_0 + \frac{\frac{2i\hbar k_0 t}{m\alpha^2} - \frac{2ix}{\alpha^2}}{1 + \frac{2i\hbar t}{m\alpha^2}}\right] - \frac{\left(x - \frac{\hbar k_0}{m}t\right)^2}{\alpha^2 + \frac{2i\hbar t}{m}} + \left(ik_0x - \frac{i\hbar k_0}{2m}t\right)$$

Now,

$$\int_{-\infty}^{\infty} dk \, e^{-\frac{\alpha^2}{4}\left(1 + \frac{2i\hbar t}{m\alpha^2}\right)\left[k - k_0 + \frac{\frac{2i\hbar k_0 t}{m\alpha^2} - \frac{2ix}{\alpha^2}}{1 + \frac{2i\hbar t}{m\alpha^2}}\right]^2} = \frac{2\sqrt{\pi}}{\alpha}\left(1 + \frac{2i\hbar t}{m\alpha^2}\right)^{-1/2}$$

So, $\psi(x,t) = \dfrac{\sqrt{\alpha}}{(2\pi)^{3/4}} \dfrac{2\sqrt{\pi}}{\alpha\left(1+\frac{2i\hbar t}{m\alpha^2}\right)^{1/2}} e^{-\frac{\left(x-\frac{\hbar k_0}{m}t\right)^2}{\alpha^2+\frac{2i\hbar t}{m}}} e^{ik_0x} e^{-\frac{i\hbar k_0}{2m}t}$

Now, $\dfrac{1}{\sqrt{1+\frac{2i\hbar t}{m\alpha^2}}} = \left(\dfrac{1}{1+\frac{2i\hbar t}{m\alpha^2}}\right)^{1/2} = \left(\dfrac{1-\frac{2i\hbar t}{m\alpha^2}}{1+\frac{4\hbar^2t^2}{m^2\alpha^4}}\right)^{1/2}$

$$\left(1-\frac{2i\hbar t}{m\alpha^2}\right)^{1/2} = \left[\left(1+\frac{4\hbar^2t^2}{m^2\alpha^4}\right)^{1/2} e^{-i\theta}\right]^{1/2} \qquad \tan\theta = \frac{2\hbar t}{m\alpha^2}$$

$$= \left(1+\frac{4\hbar^2t^2}{m^2\alpha^4}\right)^{1/4} e^{-i\theta/2}$$

So, $\psi(x,t) = \dfrac{\sqrt{\alpha}}{(2\pi)^{3/4}} \dfrac{2\sqrt{\pi}}{\alpha\left(1+\frac{4\hbar^2t^2}{m^2\alpha^4}\right)^{1/4}} e^{-i\theta/2} e^{-\frac{\left(x-\frac{\hbar k_0}{m}t\right)^2}{\alpha^2+\frac{2i\hbar t}{m}}} e^{ik_0x} e^{-\frac{i\hbar k_0}{2m}t}$

$\psi(x,t) = \left(\dfrac{2\alpha^2}{\pi}\right)^{1/4} \dfrac{e^{i\varphi}}{\left(\alpha^4+\frac{4\hbar^2t^2}{m^2}\right)^{1/4}} e^{ik_0x} e^{-\left\{\frac{\left(x-\frac{\hbar k_0}{m}t\right)^2}{\alpha^2+\frac{2i\hbar t}{m}}\right\}}$, where

$$\varphi = -\theta - \frac{\hbar k_0}{2m}t, \qquad \tan 2\theta = \frac{2\hbar t}{m\alpha^2}$$

$$|\psi(x,t)|^2 = \left(\frac{2}{\pi\alpha^2}\right)^{1/2} \frac{1}{\sqrt{1+\frac{4\hbar^2t^2}{m^2\alpha^4}}} \exp\left\{-\frac{2\alpha^2\left(x-\frac{\hbar k_0}{m}t\right)^2}{\alpha^4+\frac{4\hbar^2t^2}{m^2}}\right\}$$

So, $\Delta x(t) = \dfrac{\alpha}{2}\sqrt{1+\dfrac{4\hbar^2t^2}{m^2\alpha^4}}$.

Now, $\Delta x(t=0) = \frac{\alpha}{2}$.

$$\Delta x(t=10) = \frac{\alpha}{2}\sqrt{1+\frac{400\hbar^2}{m^2\alpha^4}}$$

$$\frac{\Delta x(t=10)}{\Delta x(t=0)} = \sqrt{1+\frac{400\hbar^2}{m^2\alpha^4}}$$

$$\text{fractional change} = \sqrt{1+\frac{400\hbar^2}{m^2a^4}} - 1$$

For $m = 1g$, $\alpha = \sqrt{2}\,10^{-9}$ we find

$$f = \left(1 + \frac{400(6.63\times10^{-34})^2}{(10^{-3})^2(4)(10^{-9})^4}\right)^{1/2} - 1$$

$$\sim 1 + \frac{1}{2}\left(\quad\right) - 1$$

$$\sim \frac{50(6.63\times10^{-34})^2}{(10^{-6})(10^{-36})} = 50(6.63)^2\,\frac{10^{-68}}{10^{-42}}$$

$$\sim 10^3\,\frac{10^{-68}}{10^{-42}}$$

$$f \sim 10^{-23}$$

(6) Calculate $[p^2, x^2]$.

Method 1

$x \to x$

$p \to \frac{\hbar}{i}\frac{\partial}{\partial x}$

$p^2 = \left(\frac{\hbar}{i}\right) \frac{\partial^2}{\partial x^2} = -\hbar^2 \frac{\partial^2}{\partial x^2}$

$$[p^2, x^2] = \left[-\hbar^2 \frac{\partial^2}{\partial x^2}, x^2\right]$$

$$[p^2, x^2] f = \left(-\hbar^2 \frac{\partial^2}{\partial x^2} x^2 + \hbar^2 x^2 \frac{\partial^2}{\partial x^2}\right) f$$

$$= -\hbar^2 \frac{\partial^2}{\partial x^2}(x^2 f) + \hbar^2 x^2 \frac{\partial^2 f}{\partial x^2}$$

$$= -\hbar^2 \frac{\partial}{\partial x}\left(2x f + x^2 \frac{\partial f}{\partial x}\right) + \hbar^2 x^2 \frac{\partial^2 f}{\partial x^2}$$

$$= -\hbar^2 \left(2f + 2x \frac{\partial f}{\partial x} + 2x \frac{\partial f}{\partial x} + x^2 \frac{\partial^2 f}{\partial x^2}\right) + \hbar^2 x^2 \frac{\partial^2 f}{\partial x^2}$$

$$= -2\hbar^2 \left(1 + 2x \frac{\partial}{\partial x}\right) f$$

$$= -2\hbar^2 f - 4\hbar^2 x \frac{i}{\hbar} \frac{\hbar}{i} \frac{\partial}{\partial x} f$$

$$= -2\hbar^2 f - 4\hbar^2 x \frac{i}{\hbar} p f$$

$$= (-2\hbar^2 - 4i\hbar\, xp) f$$

$$\therefore \quad [p^2, x^2] = -2\hbar^2 - 4i\hbar\, xp$$

Method 2

$$[x,p] = i\hbar \quad , \qquad px = xp + [p,x] = xp + \frac{\hbar}{i}$$

$$[p^2, x^2] = p^2x^2 - x^2p^2$$

$$= ppxx - xxpp$$

$$= p(xp - i\hbar)x - x(px + i\hbar)p$$

$$= pxpx - i\hbar\, px - xpxp - i\hbar\, xp$$

$$= (xp - i\hbar)(xp - i\hbar) - i\hbar(xp - i\hbar) - xpxp - i\hbar\, xp$$

$$= \cancel{xpxp} - 2i\hbar\, xp - \hbar^2 - i\hbar\, xp - \hbar^2 \cancel{- xpxp} - i\hbar\, xp$$

$$[p^2, x^2] = -2\hbar^2 - 4i\hbar\, xp$$

Method 3

$$[p^2, x^2] = p[p, x^2] + [p, x^2]p$$

$$= px[p,x] + p[p,x]x + x[p,x]p + [p,x]xp$$

$$= px(-i\hbar) + (-i\hbar)px + (-i\hbar)xp + (-i\hbar)xp$$

$$= -2i\hbar\, px - 2i\hbar\, xp$$

$$= -2i\hbar(xp - i\hbar) - 2i\hbar\, xp$$

$$[p^2, x^2] = -2\hbar^2 - 4i\hbar\, xp$$

7. $\psi(x) = Ce^{ikx} + De^{-ikx}$

$$\psi^*(x) = C^*e^{-ikx} + D^*e^{ikx}$$

$$\frac{\partial \psi}{\partial x} = ik\left(Ce^{ikx} - De^{-ikx}\right)$$

$$\frac{\partial \psi^*}{\partial x} = -ik\left(C^*e^{-ikx} - D^*e^{ikx}\right)$$

$$j(x) = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right)$$

$$= \frac{\hbar k}{2m}\left\{\left(C^*e^{ikx} + D^*e^{+ikx}\right)\left(Ce^{ikx} - De^{-ikx}\right) + \left(C^*e^{-ikx} - D^*e^{ikx}\right)\left(Ce^{ikx} + De^{-ikx}\right.\right.$$

$$= \frac{\hbar k}{2m}\left\{ |C|^2 - |D|^2 + \cancel{D^*C}\,e^{2ikx} - \cancel{C^*D\,e^{-2ikx}} + |C|^2 - |D|^2 + C^*\cancel{D\,e^{-2ikx}} - \cancel{D^*C\,e^{2i}}\right.$$

$$= \frac{\hbar k}{m}\left(|C|^2 - |D|^2\right)$$