

Quantum Physics 130A

April 1, 2006

1 HOMEWORK 1: Due Friday, Apr. 14

1. A polished silver plate is hit by beams of photons of known energy. It is measured that the maximum electron energy is 3.1 ± 0.11 eV for 2000 Angstrom light and 1.80 ± 0.07 eV for 2580 Angstrom light. Determine Planck's constant and its error based on these measurements.
2. What is the DeBroglie wavelength for each of the following particles? The energies given are the kinetic energies.
 - a 13.6 eV electron
 - a 10^5 MeV neutron
 - a 1 gram lead ball moving with a velocity of 500 m/sec.
3. A 150 keV photon collides with an electron initially at rest. The photon is observed to scatter at 90 degrees in the electron rest frame. What are the kinetic energies of the electron and photon after the scattering?
4. Use the energy density in a cavity as a function of frequency and T

$$u(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1}$$

to calculate the emissive power of a black body $E(\lambda, T)$ as a function of wavelength and temperature.

5. Use the Rydberg formula to calculate the three longest wavelengths in the Lyman series, the three longest wavelengths in the Balmer series, and the three longest in the Paschen-Bach series. For each wavelength, state what part of the EM spectrum it belongs too.
6. Light of wavelength 300 nm is incident upon two narrow slits separated by 200 microns. At what angle with the first null in the intensity distribution appear? The first intensity maximum is at zero angle. At what angle will the second maximum appear?
7. The Dirac delta function has the property that $\int_{-\infty}^{\infty} f(x)\delta(x - x_0) dx = f(x_0)$
Find the momentum space wave function $\phi(p)$ if $\psi(x) = \delta(x - x_0)$.
8. Use the uncertainty principle to estimate the energy of the ground state of a harmonic oscillator with the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$.
9. Estimate the kinetic energy of an electron confined to be inside a nucleus of radius 5 Fermis. Estimate the kinetic energy of a neutron confined inside the same nucleus.

2 Homework 2: Due Friday, April 21

1. The momentum space wave function is given by $\phi(p) = \delta(5p - \hbar k_0)$. What is the wavefunction in position space?

2. Prove that

$$\int_{-\infty}^{\infty} \psi^*(x)x\psi(x)dx = \int_{-\infty}^{\infty} \phi^*(p) \left(i\hbar \frac{\partial}{\partial p} \right) \phi(p)dp.$$

Remember that the wave functions go to zero at infinity.

3. Directly calculate the the RMS uncertainty in x for the state $\psi(x) = Ce^{\frac{-x^2}{\alpha}}$ (where C is a normalization constant to be determined by you) by computing

$$\Delta x = \sqrt{\langle \psi | (x - \langle x \rangle)^2 | \psi \rangle}.$$

4. Calculate $\langle p^n \rangle$ for the state in the previous problem. Use this to calculate Δp in a similar way to the Δx calculation.
5. Use the calculation of a spreading Gaussian wave packet to find the fractional change in size of a wave packet between $t = 0$ and $t = 10$ seconds for an electron localized to 10 Angstrom. Now find the fraction change for a 1 gram weight localized to 1 nanometer.
6. Calculate the commutator $[p^2, x^2]$. Do this by two methods showing you get the same answer. First use the differential operator for p to do the calculation, then use the basic commutator relation $px = xp + [p, x] = xp + \frac{\hbar}{i}$.
7. The wave function for a particle is initially $\psi(x) = Ce^{ikx} + De^{-ikx}$. What is the probability flux $j(x)$?

3 Homework 3: Due Friday, April 28

1. Calculate the commutators $[p, x^n]$ and $[p^m, x^n]$.
2. Prove that the parity operator defined by $P\psi(x) = \psi(-x)$ is a hermitian operator and find its possible eigenvalues.
3. A particle is in the first excited state of a box of length L . What is that state? Now one wall of the box is suddenly moved outward so that the new box has length $3L$. What is the probability for the particle to be in the ground state of the new box? What is the probability for the particle to be in the first excited state of the new box? You may find it useful to know that

$$\int \sin(Ax) \sin(Bx) dx = \frac{\sin((A-B)x)}{2(A-B)} - \frac{\sin((A+B)x)}{2(A+B)}.$$

4. Consider the functions of one angle $\psi(\theta)$ with $-\pi \leq \theta \leq \pi$ and $\psi(-\pi) = \psi(\pi)$. Show that the angular momentum operator $L = \frac{\hbar}{i} \frac{d}{d\theta}$ has real expectation values.
5. A particle is initially in the n^{th} eigenstate of a box of length $2a$. Suddenly the walls of the box are completely removed. Calculate the probability to find that the particle has momentum between p and $p + dp$. Is energy conserved?
6. A particle is in a box with solid walls at $x = \pm \frac{L}{2}$. The state at $t = 0$ is constant $\psi(x, 0) = \sqrt{\frac{2}{L}}$ for $-\frac{L}{2} < x < 0$ and the $\psi(x, 0) = 0$ everywhere else. Write this state as a sum of energy eigenstates of the particle in a box. Write $\psi(x, t)$ in terms of the energy eigenstates. Write the state at $t = 0$ as $\phi(p)$. Would it be correct (and why) to use $\phi(p)$ to compute $\psi(x, t)$?
7. A general one dimensional scattering problem could be characterized by an (arbitrary) potential $V(x)$ which is localized by the requirement that $V(x) = 0$ for $|x| > a$. Assume that the wave-function is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Ce^{ikx} + De^{-ikx} & x > a \end{cases}$$

Relating the “outgoing” waves to the “incoming” waves by the matrix equation

$$\begin{pmatrix} C \\ B \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

show that

$$\begin{aligned} |S_{11}|^2 + |S_{21}|^2 &= 1 \\ |S_{12}|^2 + |S_{22}|^2 &= 1 \\ S_{11}S_{12}^* + S_{21}S_{22}^* &= 0 \end{aligned}$$

Use this to show that the S matrix is unitary.

8. Calculate the S matrix for the potential

$$V(x) = \begin{cases} V_0 & |x| < a \\ 0 & |x| > a \end{cases}$$

and show that the above conditions are satisfied.

4 Homework 4: Due Friday, May 12

1. The odd bound state solution to the potential well problem bears many similarities to the zero angular momentum solution to the 3D spherical potential well. Assume the range of the potential is 2.7×10^{-13} cm, the binding energy is -1.1 MeV, and the mass of the particle is 940 MeV. Find the depth of the potential in MeV. (The equation to solve is transcendental.)
2. Find the three lowest energy wave-functions for the harmonic oscillator.
3. Assume the potential for particle bound inside a nucleus is given by

$$V(x) = \begin{cases} -V_0 & x < R \\ \frac{\hbar^2 \ell(\ell+1)}{2mx^2} & x > R \end{cases}$$

and that the particle has mass m and energy $E > 0$. Estimate the lifetime of the particle inside this potential.

4. In a 1D square well, there is always at least one bound state. Assume the width of the square well is a . By the uncertainty principle, the kinetic energy of an electron localized to that width is $\frac{\hbar^2}{2ma^2}$. How can there be a bound state even for small values of V_0 ?
5. At $t = 0$ a particle is in the one dimensional Harmonic Oscillator state $\psi(t = 0) = \frac{1}{\sqrt{2}}(u_1 + u_2)$. Is ψ correctly normalized? Compute the expected value of x as a function of time by doing the integrals in the x representation.
6. Prove the Schwartz inequality $|\langle \psi | \phi \rangle|^2 \leq \langle \psi | \psi \rangle \langle \phi | \phi \rangle$. (Start from the fact that $\langle \psi + C\phi | \psi + C\phi \rangle \geq 0$ for any C .)
7. The hyper-parity operator H has the property that $H^4\psi = \psi$ for any state ψ . Find the eigenvalues of H for the case that it is not Hermitian and the case that it is Hermitian.
8. Find the correctly normalized energy eigenfunction $u_4(x)$ for the 1D harmonic oscillator.
9. A beam of particles of energy $E > 0$ coming from $-\infty$ is incident upon a double delta function potential in one dimension. That is $V(x) = \alpha\delta(x + \frac{b}{2}) - \alpha\delta(x - \frac{b}{2})$.
 - a) Find the solution to the Schrödinger equation for this problem.
 - b) Determine the coefficients needed to satisfy the boundary conditions.
 - c) Calculate the probability for a particle in the beam to be reflected by the potential and the probability to be transmitted.

5 Homework 5: Due Friday, May 19

- At $t = 0$, a 1D harmonic oscillator is in a linear combination of the energy eigenstates

$$\psi = \sqrt{\frac{4}{7}}u_4 + i\sqrt{\frac{3}{7}}u_3$$

Find the expected value of p as a function of time using operator methods.

- Evaluate the “uncertainty” in x for the 1D HO ground state $\sqrt{\langle u_0|(x - \bar{x})^2|u_0\rangle}$ where $\bar{x} = \langle u_0|x|u_0\rangle$. Similarly, evaluate the uncertainty in p for the ground state. What is the product $\Delta p\Delta x$? Now do the same for the first excited state. What is the product $\Delta p\Delta x$ for this state?

- The 1D model of a crystal puts the following constraint on the wave number k .

$$\cos(\phi) = \cos(ka) + \frac{ma^2V_0 \sin(ka)}{\hbar^2 ka}$$

Assume that $\frac{ma^2V_0}{\hbar^2} = \frac{3\pi}{2}$ and plot the constraint as a function of ka . Plot the allowed energy bands on an energy axis assuming $V_0 = 2$ eV and the spacing between atoms is 5 Angstroms.

- An operator is Unitary if $UU^\dagger = U^\dagger U = 1$. Prove that a unitary operator preserves inner products, that is $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$. Show that if the states $|u_i\rangle$ are orthonormal, that the states $U|u_i\rangle$ are also orthonormal. Show that if the states $|u_i\rangle$ form a complete set, then the states $U|u_i\rangle$ also form a complete set.

$$\begin{aligned} \sum_i |u_i\rangle\langle u_i| &= 1 \\ U \sum_i |u_i\rangle\langle u_i| U^\dagger &= U1U^\dagger \\ \sum_i U|u_i\rangle\langle u_i| U^\dagger &= 1 \\ \sum_i |Uu_i\rangle\langle Uu_i| &= 1 \end{aligned}$$

- Show that if an operator H is hermitian, then the operator e^{iH} is unitary.
- Calculate $\langle u_i|x|u_j\rangle$ and $\langle u_i|p|u_j\rangle$.
- Calculate $\langle u_i|xp|u_j\rangle$ by direct calculation. Now calculate the same thing using $\sum_k \langle u_i|x|u_k\rangle\langle u_k|p|u_j\rangle$.
- If $h(A^\dagger)$ is a polynomial in the operator A^\dagger , show that $Ah(A^\dagger)u_0 = \frac{dh(A^\dagger)}{dA^\dagger}u_0$. As a result of this, note that since any energy eigenstate can be written as a series of raising operators times the ground state, we can represent A by $\frac{d}{dA^\dagger}$.

9. At $t = 0$ a particle is in the one dimensional Harmonic Oscillator state $\psi(t = 0) = \frac{1}{\sqrt{2}}(u_0 + u_1)$.
- Compute the expected value of x as a function of time using A and A^\dagger in the Schrodinger picture.
 - Now do the same in the Heisenberg picture.

6 Homework 6: Due Friday, May 26

1. The energy spectrum of hydrogen can be written in terms of the principal quantum number n to be $E = -\frac{\alpha^2 \mu c^2}{2n^2}$. What are the energies (in eV) of the photons from the $n = 2 \rightarrow n = 1$ transition in hydrogen and deuterium? What is the difference in photon energy between the two isotopes of hydrogen?
2. Prove that the operator that exchanges two identical particles is Hermitian.
3. Two identical, non-interacting spin $\frac{1}{2}$ particles are in a box. Write down the full lowest energy wave function for both particles with spin up and for one with spin up and the other spin down. Be sure your answer has the correct symmetry under the interchange of identical particles.
4. At $t = 0$ a particle is in the one dimensional Harmonic Oscillator state $\psi(t = 0) = \frac{1}{\sqrt{2}}(u_1 + u_3)$. Compute the expected value of x^2 as a function of time.
5. Calculate the Fermi energy of a gas of massless fermions with n particles per unit volume.
6. The number density of conduction electrons in copper is 8.5×10^{22} per cubic centimeter. What is the Fermi energy in electron volts?
7. The radius of a nucleus is approximately $1.1A^{\frac{1}{3}}$ Fermis, where $A = N + Z$, N is the number of neutrons, and Z is the number of protons. A Lead nucleus consists of 82 protons and 126 neutrons. Estimate the Fermi energy of the protons and neutrons separately.
8. The momentum operator conjugate to any coordinate x_i is $\frac{\hbar}{i} \frac{\partial}{\partial x_i}$. Calculate the commutators of the center of mass coordinates and momenta $[P_i, R_j]$ and of the internal coordinates and momenta $[p_i, r_j]$. Calculate the commutators $[P_i, r_j]$ and $[p_i, R_j]$.
9. A particle is in the state $\psi = R(r) \left(\sqrt{\frac{1}{3}} Y_{21} + i \sqrt{\frac{1}{3}} Y_{20} - \sqrt{\frac{1}{3}} Y_{22} \right)$. Find the expected values of L^2 , L_z , L_x , and L_y .
10. A particle is in the state $\psi = R(r) \left(\sqrt{\frac{1}{3}} Y_{11} + i \sqrt{\frac{2}{3}} Y_{10} \right)$. If a measurement of the x component of angular momentum is made, what are the possible outcomes and what are the probabilities of each?
11. Calculate the matrix elements $\langle Y_{\ell m_1} | L_x | Y_{\ell m_2} \rangle$ and $\langle Y_{\ell m_1} | L_x^2 | Y_{\ell m_2} \rangle$

7 Homework 7: Due Friday, June 2

1. The Hamiltonian for a rotor with axial symmetry is $H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_2}$ where the I are constant moments of inertia. Determine and plot the eigenvalues of H for dumbbell-like case that $I_1 \gg I_2$.
2. Prove that $\langle L_x^2 \rangle = \langle L_y^2 \rangle = 0$ is only possible for $\ell = 0$.
3. Write the spherical harmonics for $\ell \leq 2$ in terms of the Cartesian coordinates x , y , and z .
4. A particle in a spherically symmetric potential has the wave-function $\psi(x, y, z) = C(xy + yz + zx)e^{-\alpha r^2}$. A measurement of L^2 is made. What are the possible results and the probabilities of each? If the measurement of L^2 yields $6\hbar^2$, what are the possible measured values of L_z and what are the corresponding probabilities?
5. Calculate the $\ell = 0$ phase shift for a hard sphere $V = \infty$ for $r < a$ and $V = 0$ for $r > a$. What are the limits for ka large and small?

8 Homework 8: Due Friday, June 9

1. Calculate the $\ell = 0$ phase shift for the spherical potential well for both an attractive and repulsive potential.
2. An electron in the Hydrogen potential $V(r) = -\frac{e^2}{r}$ is in the state $\psi(\vec{r}) = Ce^{-\alpha r}$. Find the value of C that properly normalizes the state. What is the probability that the electron be found in the ground state of Hydrogen?
3. Calculate the difference in wavelengths of the 2p to 1s transition in Hydrogen and Deuterium. Calculate the wavelength of the 2p to 1s transition in positronium.
4. Tritium is an unstable isotope of Hydrogen with a proton and two neutrons in the nucleus. Assume an atom of Tritium starts out in the ground state. The nucleus (beta) decays suddenly into that of He^3 . Calculate the probability that the electron remains in the ground state.
5. A hydrogen atom is in the state $\psi = \frac{1}{6} (4\psi_{100} + 3\psi_{211} - \psi_{210} + \sqrt{10}\psi_{21-1})$. What are the possible energies that can be measured and what are the probabilities of each? What is the expectation value of L^2 ? What is the expectation value of L_z ? What is the expectation value of L_x ?
6. What is $P(p_z)$, the probability distribution of p_z for the Hydrogen energy eigenstate ψ_{210} ? You may find the expansion of e^{ikz} in terms of Bessel functions useful.
7. The differential equation for the 3D harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$ has been solved in the notes, using the same techniques as we used for Hydrogen. Use the recursion relations derived there to write out the wave functions $\psi_{n\ell m}(r, \theta, \phi)$ for the three lowest energies. You may write them in terms of the standard $Y_{\ell m}$ but please write out the radial parts of the wavefunction completely. Note that there is a good deal of degeneracy in this problem so the three lowest energies actually means 4 radial wavefunctions and 10 total states. Try to write the solutions ψ_{000} and ψ_{010} in terms of the solutions in cartesian coordinates with the same energy ψ_{n_x, n_y, n_z} .